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Program-iteration pattern algorithm for the elasto-plastic frictional contact FM-BEM

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Abstract: A new program-iteration pattern algorithm, incomplete generalized minimal residual method (IGMRES(m)) based on the fast multipole boundary element method (FM-BEM), was proposed for the solution of highly nonlinear equations and its convergence theory was established. With help of truncation technology, a new recursion formulae with the proposed method using only some of the calculated vectors to compute the following vectors, which could greatly reduce the computation and memory requirement. The fast multipole method (FMM) was used to compute the matrix-vector products. Numerical experiments proved that the new algorithm is highly efficient for computing elasto-plastic frictional contact problems, especially for complicated iteration and time-consuming calculation. And it can greatly reduce the iteration times and improve computational efficiency with ensured numerical accuracy.

Key words: FM-BEM (fast multipole boundary element method); IGMRES(m) (incomplete generalized minimal residual method) algorithm; FMM (fast multipole method); elasto-plastic contact with friction; truncation index

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弹塑性摩擦接触多极边界元法的规划-迭代型算法

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摘要:提出一种基于多极边界元法(FM-BEM)的规划-迭代型不完全广义极小残值法(简称 IGMRES(m)),并建立其收敛性理论.新求解算法采用截断技术,在迭代时仅使用前面计算出的部分向量构造新的递推式以计算后面的向量,矩阵和向量的乘积采用多极展开法(FMM)计算,使得计算量和存储量大为减少.通过数值试验证明,新算法可有效地处理弹塑性摩擦接触迭代的繁杂和费时问题,在确保数值计算精度的前提下,可大大减少迭代次数,显著提高计算效率.

关键词:FM-BEM;IGMRES(m)算法;FMM;弹塑性摩擦接触;截断指标

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0 Introduction

The solution of a nonlinear equation can't avoid a complicated and time-consuming iteration process. For example, the coupled governing equation for 3-D elasto-plastic frictional contact is multi-nonlinear. It can be discretized into a rate matrix equation, which is written as

$$A\dot{X} = \dot{F} \tag{1}$$

where A is a nonsingular coefficient matrix, $\dot{X}$ an unknown vector of displacement and traction, and $\dot{F}=\dot{f}(u,t)+\dot{f}(\Delta)+\dot{f}(\mu)+\dot{f}(\sigma^p)$ a known vector in which u and t indicate the displacement and traction, respectively; Δ the gap, μ the coefficient of friction, and σ^p the plastic stress.

Influenced by the physical condition of nonlinear factors $\dot{f}(\Delta)$, $\dot{f}(\mu)$ and $\dot{f}(\sigma^p)$, the solution of Eq. (1) must be subjected to a repeated iterative convergence process. Of late years, FM-BEM has been developed and it is attractive in nonlinear analysis[1]. One of the fast algorithms with high precision to solve FM-BEM Eq. (1) is the generalized minimal residual algorithm (GMRES(m))[2~4] with a restart parameter m. The algorithm is a kind of Krylov subspace method[5], which is based on an Arnoldi complete orthogonalization process when the base vectors are constructed. And the residual norm reaches the minimum in Krylov subspace[6].

However, the GMRES(m) algorithm is inconvenient in the construction of Hessenberg matrix elements and Krylov vectors because some long recursion formulae are needed. Therefore, it is necessary to resort to the popular truncation technology and establish a kind of new truncation GMRES(m) algorithm[7, 8]. Its main thought is that only some of the calculated vectors are used to construct new recursion formulae to compute the following vectors, which can greatly reduce the computation and memory requirement. This method can easily take measures to accelerate the convergent speed and quickly obtain the final results.

For the solution of elasto-plastic frictional contact problems with complicated iteration and time consuming calculation, this paper presents a kind of incomplete generalized minimal residual method (IGMRES(m)). It is the combination of the mathematical programming method[9] and the iteration method. In FM-BEM, the fast multipole method (FMM)[10] is introduced into the IGMRES(m) algorithm to accelerate the convergence. It is on the basis of the optimized GMRES(m) algorithm[9], and makes full use of the advantages of the mathematical programming method and the iteration method. So the convergent process is stable and fast with high calculating precision.

1 Fundamental formulae for the elasto-plastic frictional contact FM-BEM

For the elastic body Ω^A without consideration the body force, the boundary integral equation for the displacement speed $\dot{u}_j$ and the traction speed $\dot{t}_j$ can be written as

$$c_{ij}\dot{u}_j^A = \int_{\Gamma^A} U_{ij}^* \dot{t}_j^A \mathrm{d}\Gamma^A - \int_{\Gamma^A} T_{ij}^* \dot{u}_j^A \mathrm{d}\Gamma^A \tag{2}$$

For the elasto-plastic deformable body Ω^B, the boundary integral equation with the plastic stress rate $\dot{\sigma}_{jk}^p$ as the initial stress form can be written as[11]

$$c_{ij}\dot{u}_j^B = \int_{\Gamma^B} U_{ij}^* \dot{t}_j^B \mathrm{d}\Gamma^B - \int_{\Gamma^B} T_{ij}^* \dot{u}_j^B \mathrm{d}\Gamma^B + \int_{\Omega^B} \varepsilon_{ijk}^* \dot{\sigma}_{jk}^p \mathrm{d}\Omega^B \tag{3}$$

where U_{ij}^*, T_{ij}^*, ε_{ijk}^* are the fundamental solutions, σ_{kij}^*, T_{kij}^*, ε_{ijkl}^* are the related kernel functions, and the superscript p indicates plasticity. The FM-BEM forms of U_{ij}^*, T_{ij}^*, ε_{ijk}^* are as follows

$$U_{ij}^*(x,y) = P_{ij}(x)\left(\frac{1}{R}\right) + Q_i(x)\left(\frac{1}{R}y_j\right) \tag{4}$$

$$T_{ij}^*(x,y) = R_{ijm}(x)\left[\frac{1}{R}n_m(y)\right] + S_{im}(x)\left[\frac{1}{R}n_m(y)y_j\right] \tag{5}$$

$$\varepsilon_{ijk}^* = P_{ijk}(x)\left(\frac{1}{R}\right) + Q_{ij}(x)\left(\frac{1}{R}y_k\right) \tag{6}$$

The FM-BEM forms of σ_{kij}^*, T_{kij}^*, ε_{ijkl}^* are as follows

$$\sigma^{*}_{kij} = P_{kij}(x)\left(\frac{1}{R}\right) + Q_{ki}(x)\left(\frac{1}{R}y_j\right) \quad (7)$$

$$T^{*}_{kij}(x,y) = O_{kij}(x)\left(\frac{1}{R^3}\right) + P_{kij}(x)\left(\frac{1}{R}\right) + Q_{kij}(x)\left[\frac{1}{R}(n_k y_k + n_i y_i + n_j y_j)\right] \quad (8)$$

$$\varepsilon^{*}_{ijkl} = O_{ijkl}(x)\left(\frac{1}{R^3}\right) + P_{ijkl}(x)\left(\frac{1}{R}\right) + Q_{ijk}(x)\left(\frac{1}{R}y_l\right) \quad (9)$$

where $i,j,m,k,l=1,2,3$, $P_{ij}(x)$ and $Q_i(x)$, $R_{ijm}(x)$ and $S_{im}(x)$, $P_{ijk}(x)$ and $Q_{ij}(x)$, $P_{kij}(x)$ and $Q_{ki}(x)$, $O_{kij}(x)$, $P_{kij}(x)$ and $Q_{kij}(x)$, $O_{ijkl}(x)$, $P_{ijkl}(x)$ and $Q_{ijk}(x)$ indicate the partial derivative of $\frac{1}{R}$ with respect to x_i, respectively[12].

After discretization, Eq. (3) can be written as Eq. (1). The right term f in Eq. (1) is formed by multiple nonlinear coupling problems. It mainly consists of the traction force and displacement $f(t,u)$, the contact gap $f(\Delta)$, the frictional dissipative energy $f(\mu)$ and the elasto-plastic deformation $f(\dot{\sigma}^p)$, which can be expressed as follows

$$f = f(t,u) + f(\Delta) + f(\mu) + f(\dot{\sigma}^p) \quad (10)$$

For the simulation of nonlinear frictional contact and the elasto-plastic deformation behavior, the discriminant mode of node-to-surface frictional contact[2] is used to avoid some errors from node-to-node slip contact and to simplify the mesh division. The condition of convergence for contact judgment is without penetration.

2 Program-iteration pattern IGMRES(m) algorithm based on the FM-BEM

2.1 IGMRES(m) Algorithm Based on the FM-BEM

A kind of program-iteration pattern IGMRES (m) algorithm is proposed to well solve the elasto-plastic frictional contact problems. Compared with the GMRES (m) algorithm[2], it shows many superiorities. The difference between them is that the base vectors $\{v_j\}(j=1,2,\cdots,m)$ formed by incomplete orthogonalization, namely $Av_j(j=1,2,\cdots,m)$, are orthogonal only for IGMRES(m) algorithm at most q vectors $v_{j0},\cdots,v_j$, $j_0=\max\{1,j-q-1\}$, with $q<m$. So the base vectors are generally not orthogonal, and the IGMRES(m) algorithm can only give an approximate or quasi-minimal residual solution. Let $q_0=j_0=\max\{1,j-q-1\}$ be the truncation index for the IGMRES(m) algorithm. Detailed steps are as follows:

(Ⅰ) Initialization

Choose the step number m, and set the parameter $q(2\leqslant q\leqslant m)$ and the precision ε;

Choose the initial value $x^{(0)}=0$, and compute $r^{(0)}=f-Ax^{(0)}$, $\beta=\|r^{(0)}\|$, $v_1=r^{(0)}/\beta$, $V_1=\{v_1\}$.

(Ⅱ) Iteration

For $j=1,2,\cdots,m$

① Incomplete orthogonalization

$$\left.\begin{aligned} h_{ij} &= (Av_j, v_i)\ (i=j_0,\cdots,j), \\ \hat{v}_{j+1} &= Av_j - \sum_{i=j_0}^{j} h_{ij} v_i \end{aligned}\right\} \quad (11)$$

where the FMM is used to compute the product of a matrix and some vectors, which can reduce the memory requirement.

② Standardization

$$h_{j+1,j} = \|\hat{v}_{j+1}\|,\ v_{j+1} = \hat{v}_{j+1}/h_{j+1,j} \quad (12)$$

③ Renew the matrices V_{j+1} and $\overline{H}_j$

$$V_{j+1} = (V_j, v_{j+1}),\ \overline{H}_j = \begin{pmatrix} \overline{H}_{j-1} & h_{ij} \\ 0 & h_{j+1,j} \end{pmatrix}_{(j+1)\times j} \quad (13)$$

$\overline{H}_j$ is a strip upper Hessenberg matrix. Its nonzero elements h_{ij} are generated by Eq. (12). When $j=1$, the first column is omitted and then

$$AV_m = V_{m+1}\overline{H}_m \quad (14)$$

The FMM is used to compute the matrix-vector products, which could accelerate the solution procedure.

(Ⅲ) Solve the least squares problems

$$\|r^{(m)}\| = \min_{y_m\in C^m} \|\beta e_1 - \overline{H}_m y_m\| \quad (15)$$

to obtain y_m[12].

(Ⅳ) Construct the approximate solutions

$$x^{(m)} = x^{(0)} + V_m y_m \quad (16)$$

(Ⅴ) Compute the modular of residual vectors

$$\|r^{(m)}\| = \|f - Ax^{(m)}\| \quad (17)$$

(Ⅵ) Restart judgment

If $\| r^{(m)} \| \leqslant \varepsilon$, $x = x^{(m)}$ is the solution and stop. Otherwise, set $x^{(0)} = x^{(m)}$ and turn to Eq. (1). ε is a given convergence criterion, and can be taken as $\varepsilon = 1.0 \times 10^{-6}$.

2.2 Convergence theory for the IGMRES(m) algorithm

The IGMRES(m) algorithm is used to accelerate the convergence rate of solution procedure and to reduce the memory requirement. But at the same time, some of its main properties such as the orthogonality of the base vectors $\{v_i\}_1^m$, the minimality of the residual norms $\| r^{(m)} \|$ are lost, which are the most important references to analyze the convergence for the algorithm. In combination with the GMRES(m) algorithm, the convergence of the IGMRES(m) truncation method is studied. And the convergence theory of the IGMRES(m) algorithm is established through theoretical analysis and numerical experiments.

For convenience, the norms of residual vectors for the IGMRES(m) algorithm and the GMRES(m) algorithm are written as $r^{(m)}(IG)$ and $r^{(m)}(G)$, respectively. Define a matrix $V_{m+1} = (v_1, v_2, \cdots, v_{m+1})$.

Theorem 2.1 Suppose that V_{m+1} is fully column-ranked. The residual norms in $K_m(r^{(0)}, A)$ satisfy the following relationship

$$\| r^{(m)}(IG) \| \leqslant S(V_{m+1}) \| r^{(m)}(G) \| \tag{18}$$

where V_{m+1} is formed through an incomplete orthogonalizing process, $S(V_{m+1}) = \| V_{m+1} \| \| V_{m+1}^{+} \|$, and V_{m+1}^{+} is the generalized inverse matrix of V_{m+1}. If the IGMRES(m) algorithm breaks down at the mth step, namely $h_{m+1,m} = 0$, then $x^{(m)}(IG) = x^{*}$.

Proof According to $r^{(0)} = \beta V_{m+1} e_1$ and Eq. (14), the residual value related to $x^{(m)}(IG)$ is

$$r^{(m)}(IG) = f - Ax^{(m)}(IG) = V_{m+1}(\beta e_1 - \overline{H}_m y_m(IG))$$

So

$$\| r^{(m)}(IG) \| = \| V_{m+1}(\beta e_1 - \overline{H}_m y_m(IG)) \| \tag{19}$$

According to $\beta V_{m+1} e_1 = r^{(0)}$, there exists $\beta e_1 = V_{m+1}^{+} r^{(0)}$. Eq. (14) is fore-multiplied by V_{m+1}^{H},

$$\overline{H}_m = V_{m+1}^{+} A V_m$$

and then Eq. (19) becomes

$$\| r^{(m)}(IG) \| = \| V_{m+1}(V_{m+1}^{+} r^{(0)} - V_{m+1}^{+} A V_m y_m(IG)) \|$$

Eq. (15) is equivalent to solving a least squares problem

$$\min_{y_m \in C^m} \| V_{m+1}^{+} r^{(0)} - V_{m+1}^{+} A V_m y_m \| = \min_{z^{(m)} \in K_m(r^{(0)}, A)} \| V_{m+1}^{+} r^{(0)} - V_{m+1}^{+} A z^{(m)} \| \tag{20}$$

So, there exists

$$\begin{aligned} &\| r^{(m)}(IG) \| \leqslant \\ &\| V_{m+1} \| \| V_{m+1}^{+} r^{(0)} - V_{m+1}^{+} A V_m y_m(IG) \| = \\ &\| V_{m+1} \| \min_{z^{(m)} \in K_m(r^{(0)}, A)} \| V_{m+1}^{+} r^{(0)} - V_{m+1}^{+} A z^{(m)} \| \leqslant \\ &\| V_{m+1} \| \| V_{m+1}^{+} \| \min_{z^{(m)} \in K_m(r^{(0)}, A)} \| r^{(0)} - A z^{(m)} \| = \\ &S(V_{m+1}) \min_{z^{(m)} \in K_m(r^{(0)}, A)} \| r^{(0)} - A z^{(m)} \| = \\ &S(V_{m+1}) \| r^{(m)}(G) \| \end{aligned}$$

and Eq. (18) is proved.

If $h_{m+1,m} = 0$, then Eq. (14) becomes

$$AV_m = V_m \overline{H}_m$$

where $\overline{H}_m$ is an $m \times m$ order strip upper Hessenberg matrix without the last row in matrix $\overline{H}_m$. It means that the columns of V_m expand into an invariant subspace of A. So the eigenvalues of $\overline{H}_m$ all belong to A and $\overline{H}_m$ is nonsingular. Now

$$\begin{aligned} \| r^{(m)}(IG) \| &= \| r^{(0)} - AV_m y_m(IG) \| = \\ &\| r^{(0)} - V_m \overline{H}_m y_m(IG) \| = \\ &\| V_m(\beta e_1 - \overline{H}_m y_m(IG)) \| \end{aligned}$$

When $y_m(IG) = \beta \overline{H}_m^{-1} e_1$, there exists

$$\min_{y_m(IG) \in C^m} \| \beta e_1 - \overline{H}_m y_m(IG) \| = 0.$$

So

$$\| r^{(m)}(IG) \| = 0,$$

namely the exact solution

$$x^{(m)}(IG) = x^{(0)} + V_m y_m(IG) = x^{*}$$

is found for the IGMRES(m) algorithm. The proof is complete. □

Comment 2.1 If $q = m$, then the IGMRES(m) algorithm equals to the GMRES(m) algorithm. V_{m+1} is a theoretically standard orthogonal matrix. So $S(V_{m+1}) = 1$, and the equal sign holds in Eq. (18). According to Eq. (18), the convergence of the IGMRES(m) algorithm strongly depends on $S(V_{m+1})$. If $S(V_{m+1})$ is appropriate, then the IGMRES(m) algorithm is sure to converge, as has been demonstrated by a large number of numerical experiments.

3 Numerical example

Consider cubes A, B and C (with sides of 50 mm, 40 mm and 30mm) in frictional contact with each other. Body A is an ideal elasto-plastic one with fixed constraint on the bottom surface, and bodies B and C are elastic ones. The model and the discrete mesh are shown in Fig. 1, and the discrete data are shown in Tab. 1. For the three bodies, the Young's modulus is $E=210$ GPa, the Poisson's ratio is $\nu=0.3$, and the coefficient of friction is $\mu=0.2$. For body A, the yield limit is 260 MPa. For body C, a uniform load $p=280$ MPa is applied to the top surface. The total load is divided into six steps, and the contact tolerance is 0.001 mm.

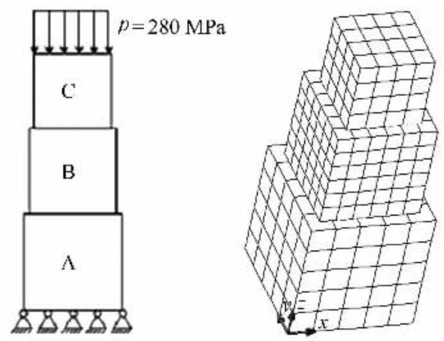

Fig. 1 Calculation model and discrete meshes

Tab. 1 Discrete data

	body A	body B	body C	sum
node number	152	218	98	468
element number	150	216	96	462
contact nodes	36	49	25	110
contact elements	25	36	16	77
interior nodes	64	0	0	64
body elements	125	0	0	125
degree number	648	801	369	1 818

The key step in the plastic analysis is the computation of the equivalent stress, which uses von Mises yield criterion to decide if the nodes are in the plastic range. The plastic stress on the node is then separated from the total stress and placed in the equation used to recompute the stress and displacement. The plastic iteration loop also includes the frictional iteration. The contact zone is fixed in each step. If penetration is discovered to have occurred during the iteration step, the step must be subdivided. The contact analysis is converged if penetration does not occur. The contact iteration also includes the plastic iteration.

The absolute convergence criterion used for the force variable is given by:

$$ABS(x_T^{t+1}-x_T^t)\leqslant\varepsilon_a$$

The relative convergence criterion used for the displacement variable is given by

$$ABS(x_U^{t+1}-x_U^t)/ABS(x_U^{t+1})\leqslant\varepsilon_r$$

When the forces and displacements are both converged, the computation is considered converged.

Set the restart parameter $m=100$, the truncation index q_0 can be taken as any integer from 1～99. When $q_0=100$, the IGMRES(m) algorithm is invalid. When $q_0=1$, it becomes the GMRES(m) algorithm. When q_0 is taken as an integer among 1～99, the iteration times and the computation time change dramatically during the solution of elasto-plastic contact with friction, as is shown in Fig. 2 and Fig. 3, respectively. From Figs. 2 and 3, the optimal truncation index $q_0=90$. The computation results are shown in Fig. 4 and Fig. 5, which are consistent with those of $q_0=1$. However, the computation time reduces from 25

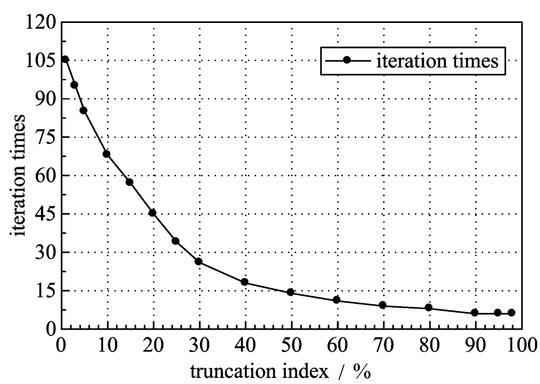

Fig. 2 Influence of truncation index on the iteration times

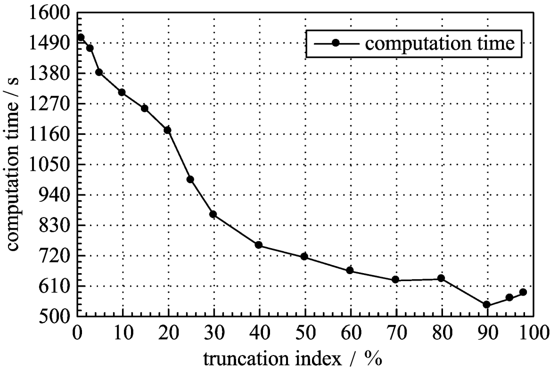

Fig. 3　Influence of truncation index on the computation time

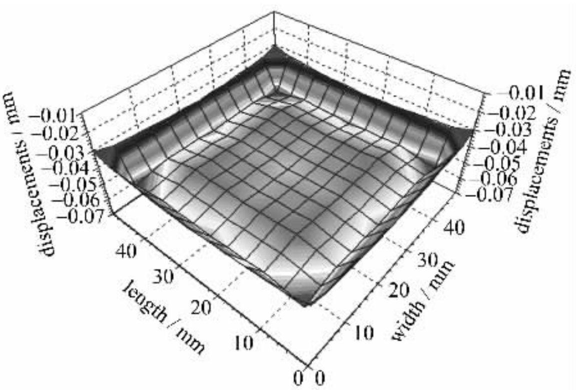

Fig. 4　Displacement distributions on the contact surface for body A

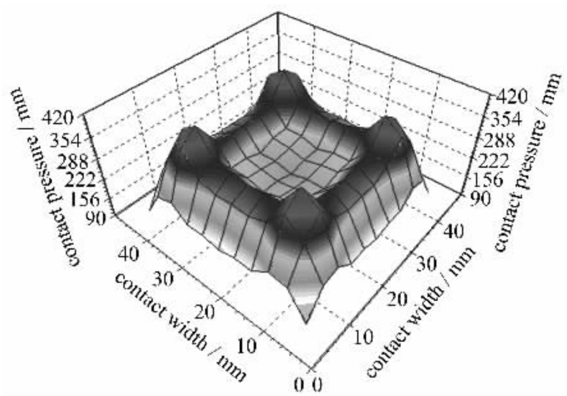

Fig. 5　Pressure distributions in contact zone

minutes and 6 seconds to 8 minutes and 59 seconds, and the iteration times reduces from 105 to 6. The efficiency can be improved by 64%, and the iteration times can be reduced by 94%, which shows the high efficiency of the new algorithm.

On the other hand, the influence of truncation index q_0 on the calculating precision is shown in Fig. 6 and Fig. 7, respectively.

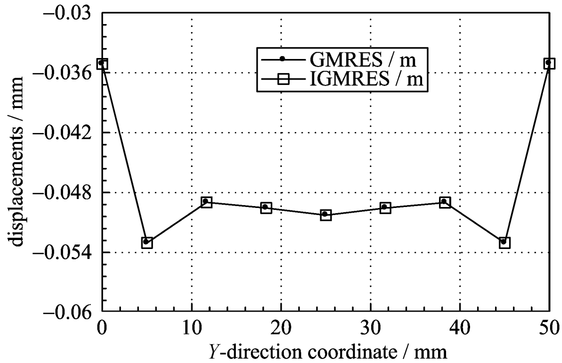

Fig. 6　Comparison of displacements on the contact surface for body A (x=11.67 mm, z=50 mm)

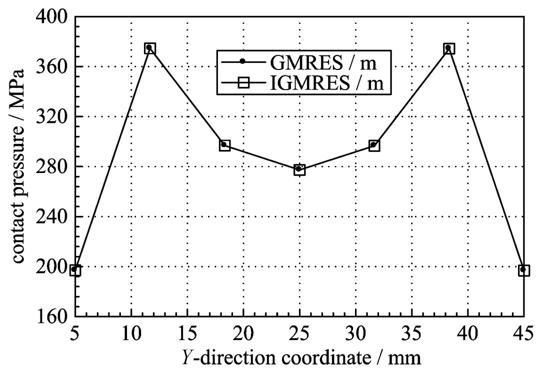

Fig. 7　Comparison of contact pressure (x=11.67 mm, z=50 mm)

Comprehensively consider calculating precision and computational efficiency, the truncation index q_0 can be chosen as 90 to dramatically improve computational efficiency with ensured calculating precision. In addition, other numerical examples also prove the high efficiency of the newly proposed algorithm with appropriate truncation index.

4　Conclusion

By using the truncation technology, a kind of new program-iteration pattern IGMRES(m) algorithm based on the FM-BEM was proposed to greatly reduce the computation and memory requirement. In combination with the GMRES(m) algorithm, the convergence theory was established for the new IGMRES(m) algorithm. Trough a numerical experiment, the high efficiency of the new algorithm and its excellent convergence were analyzed. It shows that the truncation index plays an important role for computational efficiency and calculating precision. If it is chosen properly, computational efficiency can be greatly improved and iteration times can be dramatically reduced. This algorithm is especially suitable for the solution of elasto-plastic frictional contact problems with complicated iteration and time-consuming calculation.

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