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# 均值变点估计的强相合性

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**摘要**:在二阶矩存在下,对于独立序列,考虑了均值变点的累积和(cumulative sum,CUSUM)型估计,通过截尾,证明了估计是强相合的,改进了已知的弱相合结果.作为非独立的情况,考虑了在可靠性理论、渗透理论以及多元统计分析中有着广泛应用的负相关序列,采用了另外一种截尾方法,也证明了均值变点估计的强相合性.

**关键词**:均值变点;累积和;负相关;截尾;强相合

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# Strong consistence of estimator for the change point in mean

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**Abstract**: In the case that the second moment exited, the cumulative sum (CUSUM) estimator of the change point in mean for a independent sequence was studied. Through truncation, it was proved that the estimator is strongly consistent, which improves the known consistent results. Furthermore, as a non-independent case, a negative association (NA) sequence was studied, which is widely applied in reliability, percolation theory and multivariate statistical analysis. Through another means of truncation, strong consistence for the change-point estimator was proved.

**Key words**: change point in mean; cumulative sum; negative association; truncation; strong consistent

## 0 引言

关于变点的检测已经被广泛应用于质量控制、地震灾害预测等领域中.目前关于均值变点的检测大致有两种方法:局部比较方法(可参考文献[1～5])以及CUSUM方法(主要参考文献[8]).目前在二阶矩的条件下,对于独立的样本,人们大多研究了变点估计的弱相合性,并获得了一些弱收敛速度,这对研究变点估计的渐进分布无疑是有很大作用的.然而,在实际情况中我们还需要考虑变点估计的强相合性,进一步提高估计的效果.在二阶矩存在下,文献[9,14]均获得了弱相合结果,我们在独立情形下把弱相合改进为强相合.对于相依序列,目前主要研究是关于长程相依与短程相依[11,12].事实上其他相关序列也值得探讨,比如正相关(positive association, PA)序列与负相关(negative association, NA)序列,

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有关这类序列的研究可参考文献[6,13,15]. 还有其他混合序列,如 $\rho$ 混合序列等. 在这些相依序列中,由于考虑到负相关序列的重要用途,我们考虑了负相关序列中的均值变点. 值得说明的是,其他相依情形,在附加适当的条件下,均可由本文类似得出相同的结果.

## 1 引理与主要结果

我们先介绍负相关序列的定义.

**定义 1.1** 称随机变量 $X_1,\cdots,X_n(n\geqslant 2)$ 称为 NA,如果对于 $\{1,\cdots,n\}$ 的任何两个不相交的非空子集 $A_1$ 和 $A_2$,都有

$$\mathrm{Cov}(f_1(X_i,i\in A_1),f_2(X_j,j\in A_2))\leqslant 0.$$

式中,$f_1$ 和 $f_2$ 是任何使上述协方差存在的对每个变元均非降(均非升)的函数. 称随机变量序列 $X_n(n\geqslant 1)$ 是 NA 的,如果对任何自然数 $n\geqslant 2$,$X_1,\cdots,X_n$ 都是 NA 的.

NA 序列有很多性质与独立序列相似,其中苏淳等[6]获得了 NA 序列的 Rosenthal 型不等式如引理 1.3.

**引理 1.2**(Rosenthal inequality) 若 $X_i$ 是独立的且期望为 0,$r\geqslant 2$,则

$$E\mid\sum_{i=1}^n X_i\mid^r\leqslant C_r((\sum_{i=1}^n EX_i^2)^{r/2}+\sum_{i=1}^n E\mid X_i\mid^r).$$

证明可参考文献[17].

**引理 1.3** 若 $X_i$ 是零均值的 NA 序列,$r\geqslant 2$,$\beta_r=\sup_j E\mid X_j\mid^r<\infty$. 记 $S_{a,k}=\sum_{j=0}^{k-1}X_{a+j}$,$S_{1,k}=S_k$,则存在仅与 $r$ 有关的常数 $C_r>0$,使对任何自然数 $a$ 和 $n$,有

$$E\max_{1\leqslant k\leqslant n}\mid S_{a,k}\mid^r\leqslant C_r(n\beta_r+(n\beta_2)^{r/2}).$$

证明可参考文献[6].

考虑均值不变的序列 $(X_i)_{i=1,\cdots,n}$,假设 $(Y_i)_{i=1,\cdots,n}$ 是对应的有均值变化的序列,定义为

$$Y_i=\begin{cases}X_i+\delta, & \text{if } i\leqslant k^*;\\ X_i, & \text{if } i>k^*.\end{cases}\tag{1}$$

式中,$k^*$ 为变点,$\tau^*=k^*/n$ 为变点的位置. 假设满足 $0<\gamma_1<\tau^*<\gamma_2<1$. 变点 $k^*$ 以及变点位置 $\tau^*$ 的 CUSUM 估计分别为

$$\left.\begin{aligned}\hat{k}&=\min(\arg\max_{1\leqslant k<n}\{\mid U_k\mid\}),\\ \hat{\tau}&=\hat{k}/n.\end{aligned}\right\}\tag{2}$$

式中,

$$U_k=\left(\frac{k(n-k)}{n}\right)^{1-\alpha}\left(\frac{1}{k}\sum_{i=1}^k Y_i-\frac{1}{n-k}\sum_{i=k+1}^n Y_i\right),\tag{3}$$

$\alpha$ 满足 $0\leqslant\alpha<1$.

在二阶矩存在的条件下,关于变点估计普遍获得如下的弱相合结果:如果序列 $Y_1,\cdots,Y_n$ 是独立的,则

$$P\{\mid\hat{\tau}-\tau^*\mid>\varepsilon\}\leqslant\begin{cases}\dfrac{C}{\varepsilon^2}\cdot n^{-1}, & \text{if } \alpha<1/2;\\ \dfrac{C}{\varepsilon^2}\cdot n^{-1}\log n, & \text{if } \alpha=1/2;\\ \dfrac{C}{\varepsilon^2}\cdot n^{2\alpha-2}, & \text{if } \alpha>1/2.\end{cases}\tag{4}$$

从以上可以看出最快的速度为 $n^{-1}$,对应为 $0\leqslant\alpha<1/2$.

文献[9]在二阶矩存在的条件下,假设 $Y_1,\cdots,Y_{k^*}\sim\nu_1$,$Y_{k^*+1},\cdots,Y_n\sim\nu_2$,$\nu_1\neq\nu_2$,获得了如下结果:

$$\hat{\tau}-\tau^*=O_p\left(\frac{1}{n}\right).\tag{5}$$

然而以上结果无法证明变点估计的强相合性. 在这里我们采用截尾方法来提高估计的速度. 这里只考虑 $\alpha=0$ 的情形,其他情形可类似推出.

本文中 $C$ 表示一个不依赖于 $n$ 变化的正常数. 考虑均值变化的序列 $Y_1,\cdots,Y_n$,其中不妨假设 $EX_i=0$,$1\leqslant i\leqslant n$,$\delta>0$,我们获得了如下结果:

**定理 1.4** 若序列 $X_1,\cdots,X_{k^*}$ i. i. d.,$X_{k^*+1},\cdots,X_n$ i. i. d.,且 $E|X_i|^2\leqslant C<\infty$,$i=1,\cdots,n$,则 $\hat{\tau}\to\tau^*$,a. s..

**证明** ∵

$$\begin{aligned}\mid U_k\mid-\mid U_{k^*}\mid&\leqslant\mid U_k-EU_k\mid+\\ &\mid U_{k^*}-EU_{k^*}\mid+\mid EU_k\mid-\mid EU_{k^*}\mid\leqslant\\ &2\max_{1\leqslant k<n}\mid U_k-EU_k\mid+\mid EU_k\mid-\mid EU_{k^*}\mid.\end{aligned}\tag{6}$$

把 $k$ 换成 $\hat{k}$,并注意到式(2),于是

$$\begin{aligned}\mid EU_{k^*}\mid-\mid EU_{\hat{k}}\mid&\leqslant 2\max_{1\leqslant k<n}\mid U_k-EU_k\mid\leqslant\\ &2\max_{1\leqslant k<n}\mid\sum_{i=1}^k(Y_i-EY_i)\mid+\\ &2\max_{1\leqslant k<n}\mid\sum_{i=k+1}^k(Y_i-EY_i)\mid=\\ &2\max_{1\leqslant k<n}\mid\sum_{i=1}^k X_i\mid+\end{aligned}$$

$$2\max_{1\leqslant k<n}\left|\sum_{i=k+1}^{k}X_i\right| (X_i=Y_i-EY_i)\hat{=} T_1+T_2. \qquad (7)$$

现在我们处理不等式(7)的左边，

$$EU_k=\begin{cases}\dfrac{k(n-k^*)}{n}\delta,\ k<k^*;\\ \dfrac{k^*(n-k)}{n},\ k>k^*.\end{cases}$$

$\because 0<\gamma_1<\tau^*<\gamma_2<1$，记 $\gamma=\min\{\gamma_1,1-\gamma_2\}$．于是

$$|EU_{k^*}|-|EU_{\hat{k}}|\geqslant|\hat{k}-k^*|\delta\gamma. \qquad (8)$$

由式(7)与式(8),我们可得

$$n|\hat{\tau}-\tau^*|\delta\gamma\leqslant T_1+T_2. \qquad (9)$$

因此

$$P(|\hat{\tau}-\tau^*|>\varepsilon)=P(n|\hat{\tau}-\tau^*|\delta\gamma>n\varepsilon\delta\gamma)\leqslant P\left(T_1>\frac{1}{2}n\varepsilon\delta\gamma\right)+P\left(T_2>\frac{1}{2}n\varepsilon\delta\gamma\right). \qquad (10)$$

记 $Z_i=X_iI(|X_i|<n)$,有

$$P(T_1>\frac{1}{2}n\varepsilon\delta\gamma)\leqslant P(\bigcup_{i=1}^{n}(|X_i|>n))+P(\max_{1\leqslant k<n}\left|\sum_{i=1}^{k}Z_i\right|\geqslant\frac{1}{2}n\varepsilon\delta\gamma)\hat{=} I_1+I_2,$$

$$I_1\leqslant n(P(|X_1|>n)+P(|X_n|>n)). \qquad (11)$$

$$\sum_{n=1}^{\infty}nP(|X_1|>n)\leqslant \sum_{n=1}^{\infty}n\sum_{k=n}^{\infty}EI(k\leqslant|X_1|<(k+1))\leqslant \sum_{k=1}^{\infty}EI(k\leqslant|X_1|<(k+1))\sum_{n=1}^{k}n\leqslant \sum_{k=1}^{\infty}Ck^2EI(k\leqslant|X_1|<(k+1))\leqslant \sum_{k=1}^{\infty}CE|X_1|^2I(k\leqslant|X_1|<(k+1))\leqslant CE|X_1|^2<\infty. \qquad (12)$$

类似 $\sum_{n=1}^{\infty}nP(|X_n|>n)<\infty$.

$$\therefore\ \sum_{n=1}^{\infty}P(\bigcup_{i=1}^{n}(|X_i|>n))<\infty. \qquad (13)$$

$\forall i\in\mathbb{N}$，$EX_i=0,EX_i^2<\infty$,

$$|nEZ_i|=|nEX_iI(|X_i|<n)|=|nEX_iI(|X_i|>n)|\leqslant E|X_i|^2I(|X_i|>n)\to 0\ (n\to\infty).$$

当 $n$ 充分大时,

$$I_2\leqslant P\left(\max_{1\leqslant k<n}\left|\sum_{i=1}^{k}(Z_i-EZ_i)\right|+\max_{1\leqslant k<n}\left|\sum_{i=1}^{k}EZ_i\right|\geqslant\frac{1}{2}n\varepsilon\delta\gamma\right)\leqslant P\left(\max_{1\leqslant k<n}\left|\sum_{i=1}^{k}(Z_i-EZ_i)\right|+\sum_{i=1}^{n}|EZ_i|\geqslant\frac{1}{2}n\varepsilon\delta\gamma\right)\leqslant P\left(\max_{1\leqslant k<n}\left|\sum_{i=1}^{k}(Z_i-EZ_i)\right|+n(|EZ_1|+|EZ_n|)\geqslant\frac{1}{2}n\varepsilon\delta\gamma\right)\leqslant P\left(\max_{1\leqslant k<n}\left|\sum_{i=1}^{k}(Z_i-EZ_i)\right|\geqslant\frac{1}{4}n\varepsilon\delta\gamma\right)\leqslant C\frac{E\max_{1\leqslant k<n}\left|\sum_{i=1}^{k}(Z_i-EZ_i)\right|^t}{n^t},t>2. \qquad (14)$$

由 Marcinkiewice-Zygmund 不等式,

$$I_2\leqslant C\frac{E\left|\sum_{i=1}^{n}(Z_i-EZ_i)\right|^t}{n^t},t>2. \qquad (15)$$

由引理 1.2 以及 C-r 不等式,

$$I_2\leqslant C\frac{(\sum_{i=1}^{n}EZ_i^2)^{t/2}+\sum_{i=1}^{n}E|Z_i|^t}{n^t}\leqslant C(n^{-t/2}+n^{1-t}(E|Z_1|^t+E|Z_n|^t)),\ t>2. \qquad (16)$$

因为 $t>2$,

$$\sum_{n=1}^{\infty}n^{-t/2}<\infty. \qquad (17)$$

$\forall i\in\mathbb{N}$,

$$\sum_{n=1}^{\infty}n^{1-t}E|Z_i|^t=\sum_{n=1}^{\infty}n^{1-t}E|X_i|^tI(|X_i|<n)=\sum_{n=1}^{\infty}n^{1-t}\sum_{k=1}^{n}E|X_i|^tI(k\leqslant|X_i|<k+1)=\sum_{k=1}^{\infty}E|X_i|^tI(k\leqslant|X_i|<k+1)\sum_{n=k}^{\infty}n^{1-t}\leqslant \sum_{k=1}^{\infty}k^{2-t}E|X_i|^tI(k\leqslant|X_i|<k+1)\leqslant \sum_{k=1}^{\infty}E|X_i|^2I(k\leqslant|X_i|<k+1)\leqslant CE|X_i|^2<\infty. \qquad (18)$$

由式(11)～(18),

$$\sum_{n=1}^{\infty} P\left(T_1 \geqslant \frac{1}{2} n \varepsilon \delta \gamma\right) < \infty, \tag{19}$$

类似可证

$$\sum_{n=1}^{\infty} P\left(T_2 \geqslant \frac{1}{2} n \varepsilon \delta \gamma\right) < \infty. \tag{20}$$

因此，

$$\sum_{n=1}^{\infty} P(|\hat{\tau} - \tau^*| > \varepsilon) < \infty, \tag{21}$$

由 Borel-Cantelli 引理，$\hat{\tau} \to \tau^*$，a. s. ($n \to \infty$).

**定理 1.5** 若序列 $X_1, \cdots, X_n$ 是 NA 的，$X_1, \cdots, X_{k^*}$ 同分布，$X_{k^*+1}, \cdots, X_n$ 同分布且 $E|X_i|^2 \leqslant C < \infty, i=1, \cdots, n$，则 $\hat{\tau} \to \tau^*$，a. s..

**证明** 前部分证明与式(6)～(10)相同，后部分修改如下.

令

$$Z_i'(n) = Z_i' = -nI(X_i < -n) + X_i I(|X_i| \leqslant n) + nI(X_i > n),$$

不难证明，

$$\forall i \in \mathbb{N}, \quad |nEZ_i'| \to 0, n \to \infty. \tag{22}$$

修改式(19)为

$$P\left(T_1 > \frac{1}{2} n \varepsilon \delta \gamma\right) \leqslant P\left(\bigcup_{i=1}^{n} (|X_i| > n)\right) + P\left(\max_{1 \leqslant k < n} \left|\sum_{i=1}^{k} Z_i'\right| \geqslant \frac{1}{2} n \varepsilon \delta \gamma\right). \tag{23}$$

由式(11)～(12)，$\sum_{n=1}^{\infty} P\left(\bigcup_{i=1}^{n} (|X_i| > n)\right) < \infty$.

$$E|Z_i'|^2 \leqslant C(E|X_i|^2 I(|X_i| < n) + n^2 EI(|X_i| > n)) \leqslant C\left(E|X_i|^2 + n^2 E\frac{|X_i|^2}{n^2}\right) \leqslant CE|X_i|^2 < C < \infty. \tag{24}$$

由于 $Z_i'$ 满足单调性，于是由文献[6]，$Z_i'$ 仍为 NA 变量，由引理 1.3、C-r 不等式以及式(24)，有

$$P\left(\max_{1 \leqslant k < n} \left|\sum_{i=1}^{k} Z_i'\right| \geqslant \frac{1}{2} n \varepsilon \delta \gamma\right) \leqslant C\frac{E\max_{1 \leqslant k < n} \left|\sum_{i=1}^{k} Z_i'\right|^t}{n^t} \leqslant C\frac{(n\sup_{1 \leqslant j < n} E|Z_i'|^2)^{t/2} + n\sup_{1 \leqslant j < n} E|Z_i'|^t}{n^t} \leqslant C\left[n^{-t/2} + \frac{n\sup_{1 \leqslant j < n} E|Z_j'|^t}{n^t}\right], t > 2. \tag{25}$$

由于

$$\frac{n\sup_{1 \leqslant j < n} E|Z_j'|^t}{n^t} \leqslant Cn^{1-t}(E|X_1|^t I(|X_1| < n) + E|X_n|^t I(|X_n| < n)) + Cn(EI(|X_1| > n) + EI(|X_n| > n)). \tag{26}$$

由式(12)和式(18)，有

$$\sum_{n=1}^{\infty} n \sup_{1 \leqslant j < n} E|Z_j'|^t / n^t < \infty.$$

显然 $t > 2$，$\sum_{n=1}^{\infty} n^{-t/2} < \infty$. 因此，

$$\sum_{i=1}^{\infty} P\left(\max_{1 \leqslant k < n} \left|\sum_{i=1}^{k} Z_i'\right| \geqslant \frac{1}{2} n \varepsilon \delta \gamma\right) < \infty.$$

因此，

$$\sum_{n=1}^{\infty} P(|\hat{\tau} - \tau^*| > \varepsilon) < \infty. \tag{27}$$

由 Borel-Cantelli 引理，$\hat{\tau} \to \tau^*$，a. s. ($n \to \infty$).

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③ 若 $z=[0^{n-p^{km}}]$,则 $\mathrm{Width}(x)=\mathrm{Width}(y)$;

④ 若 $z\neq[0^{n-p^{km}}]$,则

$$\mathrm{Width}(x)=p^{km}+\mathrm{Width}(z).$$

**证明**　由定义 1.2 可知结论①和②是显然的;下证③和④:

③ 若 $z=W^{p^{km}}(\boldsymbol{x})=[0^{n-p^{km}}]$,则 $\mathrm{Width}(\boldsymbol{x})\leqslant p^{km}<n$,由引理 2.4 有 $\mathrm{Width}(\boldsymbol{x})=\mathrm{Width}(\boldsymbol{y})$;

④ 由 $z=W^{p^{km}}(\boldsymbol{x})\neq[0^{n-p^{km}}]$知:$\mathrm{Width}(\boldsymbol{x})>p^{km}$,根据定义 1.2 可得

$$\mathrm{Width}(\boldsymbol{x})=p^{km}+\mathrm{Width}(z).\qquad\square$$

与算法 2.3 相比,算法 2.5 不需要进行逐步递归,减小了计算机在计算过程中所耗用的存储空间和运行时间,其计算码字广度的速度要比算法 2.3 快得多,极大地降低了运算成本,并且提高了运算的效率.

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