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Geometry and Problem Solving

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Abstract

This paper identifies a specific contents-based strategy for problem solving based on analytical geometry procedures. Here, an appropriate methodology for putting the strategy into practice, will be exposed.

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1 Introduction

Problem solving plays a crucial role in the learning of mathematics. Typically the process of problem solving combines knowledge and heuristics with specific strategies for collecting, organizing and treating information, making use of different representations, mathematical models and conversions from one language to another and establishing relationships between the learned contents. The research papers [1, 4, 6], reveal several aspects of problem solving present in different teaching-learning strategies which are of a general nature and focus on the study of the logical-economic forms of thinking, without attempting to express the specificities of the processes in their relationship with a certain content. The research carried out on the use of specific content-dependent strategies as an approach to problem solving is yet in its beginning.

Each disciplinary field has its own characteristics, the uniqueness of which means that their treatment requires particular demonstrative procedures, ways of thinking and problem solving processes that involve the specific contents of each particular field. To deal with a specific problem it is therefore often necessary to break down the barrier of general strategies and turn to particular strategies.

2 Preliminary Notes

We could mention the studies carried out by P. Ruesga and J. M. Sigarreta [5, 8, 9] which present a clear preference for specific strategies for problem solving according to specific contents. Many of the determining factors of problem solving skills are related to cognitive processes [2, 7]. It is obvious that to be successful in the solving of mathematical problems a student must be able to understand and interpret the mathematical relationships involved; but, an effective resolution of the problem is also dependent upon the student's knowledge of specific situations, ie of its contents and the way the student organizes his/her knowledge for that particular situation and the specific strategies corresponding to those contents. Authors such as Hinsley, Hayes and Simon have provided evidence to show that those who are competent in the solving of mathematical problems have a wide knowledge of problems type and the specific strategies required to solve them. The choice of a specific strategy for solving problems according to its specific contents is not incompatible with the general strategies. On the contrary, specific strategies arise naturally within any general strategy.

3 Main Results

Problem solving is a process of reasoning and, as such, consists of deduction's sequences in which, for some cases, the focus is either on the data, or on the initial conditions or in the causes, and the aim is the founding of solutions, final conditions or effects. In other cases, the reasoning deductions are made from the conclusions to the hypothesis.

The two types of routes intermingle continually such that whoever is reasoning is not necessarily consciously of doing so. Following this path, the person solving the problem brings in concepts and methods which, rightly or wrongly, he/she links with the problem through such resources as analogy and devises various problem solving processes. The strategy which we describe brings out the importance of various steps that take place when one is trying to solve a problem using the tools of Analytical Geometry. However, the bi-directionality with which the reasoning processes occur explains why the various stages that are highlighted in the strategy do not constitute a temporal sequence. Even though they may be different, some stages may occur together on more than one occasion. The characteristics of this specific field of knowledge involve the creation of an appropriate system of representation that does however depend on the nature of the problem: on the type of relationships which it expresses or on the geometric figures that it evokes or states, and that may be conceived from the start of the process or after an initial review stage and recognition of the conditions expressed in the problem. The proposed strategy consists of five actions. These are:

- *Identification*: The analysis of the relationships and of the data which the problem expresses serves to determine whether the problem can be approached using the tools of analytical geometry.

- *Selection of a system of coordinates*: In this step, the choice of the system of coordinates will be made according to the conditions of the problem. It may be rectangular, oblique or polar.

- *Representation*: The essential part of this step is always to place the figure, without losing generality, in the simplest position, so as to facilitate the calculations performed later.

- *Assignment of literal coordinates to the related elements*: The most important part of this step is to assign the coordinates to the elements of the figure in the most general way possible, making sure that the lowest possible number of variables is introduced in order to facilitate calculations.

- *Choice of elements as ordered pairs and/or vectors*: In this action it is essential to make a correct interpretation of the problem and to analyze the contents related to it having in view the choice of the necessary means to solve it. It is all about making a good choice.

Some of the given examples involve known properties covered in mid-level courses using metric geometry. Others, were originally devised to be solved by applying other contents. In the given solutions, the various stages of the described strategy are identified.

Problem 1: Prove that for any triangle ABC, $|BC|^2 = |AC|^2 + |AB|^2 - AC \cdot AB \cdot \cos(\alpha)$, where α is the angle formed by the sides AC and AB.

Solution: The property that has to be proved is metric and envolves relationships relating line segment measurements. Cartesian coordinates of the points involved, offers a possibility of converting these measurements in algebraic calculations (*Identification*), being in this case vertices of a triangle to which the generic labels A, B and C are assigned. The representation of any triangle in a coordinated system simplifies the calculations when one of the vertices is represented in the origin of the coordinates, another on the X axis and the third in a generic position (*choice of a coordinate system and representation of the figure on this system*). Accordingly, one vertex, say A is represented by $(0, 0)$. We relabel this vertex by O. Another, say C, has coordinates $(b, 0)$, and the third point, B, has generic plane coordinates (c, d). Now, the data provided is the angle α of sides AB and AC, It is therefore more economic to relate the coordinates of B to the distance $|AB|$, which we denote by a, and α. The coordinates of B are therefore $(a \cdot cos(\alpha); a \cdot sen(\alpha))$ (*assignment of literal coordinates to the related elements and choice of the elements as ordered pairs and/or vectors*). This representation makes possible to calculate the measurement of sides AC and AB by means of simplified general expressions: b and a respectively. The numeric expression obtained for the distances between points through their cartesian coordinates provides, immediately, the result.

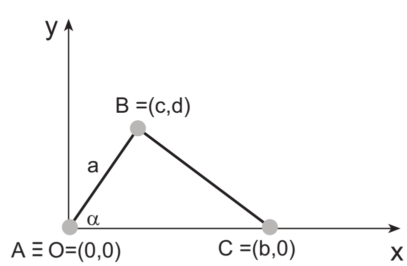

Figure 1: Cartesian representation of ABC.

$$|BC| = \sqrt{(a \cdot \cos(\alpha) - b)^2 + (a \cdot \sin(\alpha))^2},$$
$$|BC|^2 = (a)^2 + (b)^2 - 2a \cdot b \cdot \cos(\alpha),$$
$$|BC|^2 = |AC|^2 + |AB|^2 - AC \cdot AB \cdot \cos(\alpha).$$

Problem 2: Prove that three times the distance from the barycentre of a triangle to a straight line that does not intersect the sides of the triangle is equal to the sum of the distances from the vertices to that straight line.

Solution: The knowledge of the subject should tell us that if we know the cartesian coordinates $A(x_1; y_1)$, $B(x_2; y_2)$ and $C(x_3; y_3)$ of the vertices of the triangle, the corresponding cartesian coordinates of its barycentre are given by $G(\frac{x_1+x_2+x_3}{3}; \frac{y_1+y_2+y_3}{3})$ (*identification and choice of elements as ordered pairs and/or vectors*). Assuming this labeling (*assignment of literal coordinates to related elements*) let us look for a convenient reference system: undoubtedly the most convenient way of measuring the distances from points to straight lines is obtained when the straight line is one of the coordinate axis say, the X axis. Let assume, without lost of generality, the X axis as the straight line mentioned in the text (*choice of a system of coordinates*). The distance from the barycentre to the X axis is therefore given by its y-coordinate (*representation*).

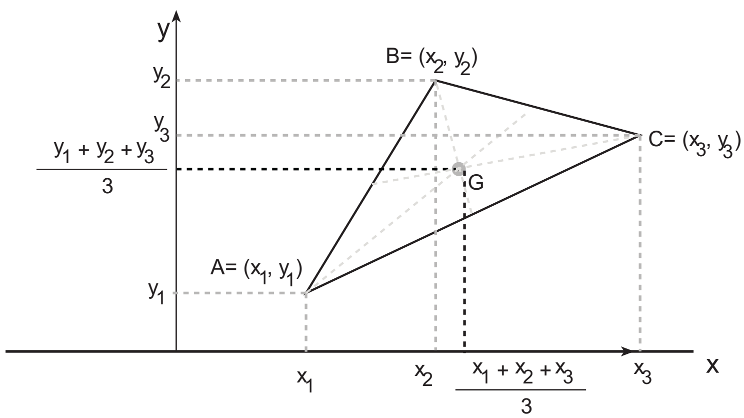

Figure 2: Cartesian representation.

Within this representation, we need only to prove that three times the y-coordinate of the barycentre equals the sum of the distances from each of the vertices to the X axis, and this is exactly the sum d of the y-coordinates of the three points, $d = y_1 + y_2 + y_3$. Since the distance from the barycentre to the same straight line $y = 0$ is given by $d_1 = \frac{y_1+y_2+y_3}{3}$, $d = 3d_1$, which is what we set out to prove.

Problem 3: A square and an equilateral triangle are inscribed in a circumference of unitary radius and one of the vertices of the triangle coincides with one of the vertices of the square. Find the common area to the triangle and to the square.

Solution: The geometric figures mentioned in the body of the problem are regular ones. Furthermore, a circumference $\mathcal{C}$ of radius 1 can be expressed by an analytical equation in a cartesian system, (*identification*) which has a simplified form when the origin of the coordinate system coincides with its centre. This choice for the representation of $\mathcal{C}$ provides further information

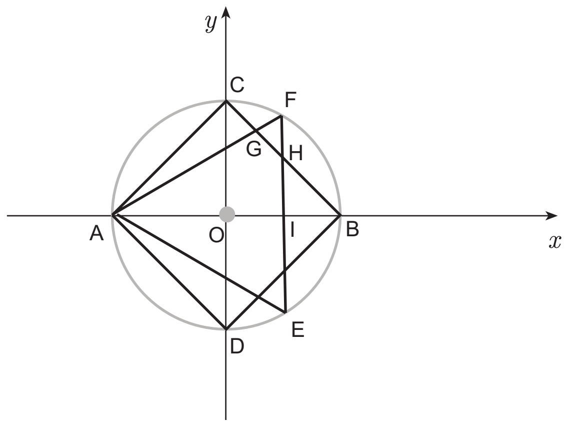

Figure 3: Cartesian representation.

regarding the points of intersection with the coordinated axes. If, in addition, we make use of the perpendicularity of the coordinated axes to represent the diagonals of the square, the common vertex A be $(-1, 0)$ and the X-axis to be a symmetry axis for the given configuration, which does not mean any loss of generality, we have choosen a system of coordinates that as we shall see will reveal the solution of the problem is a quite simplified way, (*choice of a coordinate system and representation of the figure on the system*). In Figure 3 we illustrate the configuration obtained , using the chosen system of coordinates where some important points are indicated (*choice of the elements as ordered pairs and/or vectors*).

Due to the symmetry of the figure, the area $A^\star$ that we are asked to calculate is given by: $A^\star = 2(A_{AGB} - A_{HIB})$. As the circumference has a unitary radius and it has its centre in the origin of the coordinates, its equation is $x^2 + y^2 = 1$. As the X-axis bisects the angle EAF the equation of the straight line going through A and F is $y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$, besides the equation of the straight line going through B and C is, $y = x + 1$. From these two equations we can calculate the coordinates of the points G, F and H which are: $G = (2 - \sqrt{3}; \sqrt{3} - 1)$, $F = (\frac{1}{2}; \frac{\sqrt{3}}{2})$ and $H = (\frac{1}{2}; \frac{1}{2})$. Accordingly, $A^\star = \frac{8\sqrt{3}-9}{4}$.

Problem 4: Let ABC be a triangle, G its barycentre and P any point of the plane. Prove that: $3|PG|^2 = |AP|^2 + |BP|^2 + |CP|^2 - 1/3(|AB|^2 + |BC|^2 + |AC|)^2$.

Solution: As seen in Problem 2, the assignment of generic cartesian coordinates to the vertices of the triangle provides a relation for the cartesian coordinates of the barycentre (*identification, choice of elements as ordered pairs and assignation of literal coordinates to the related elements*). Let us choose a rectangular coordinate system taking the vertex A to be the origin of the system, the x-axis the axis defined by the points A and B oriented from A to B. The y axis be ortogonal to X- axis and oriented in a way that the triangle ABC belongs to the upper closed half-plane, see Figure 3,(*choice of a system of coordinates and representation of the figure on the system of coordinates*).

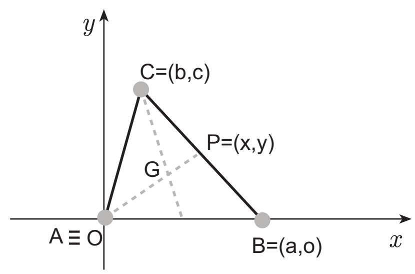

Figure 4: Cartesian representation.

In accordance to the chosen system A has coordinates $(0,0)$, $B=(a,0)$ for some positive real a, $C(b,c)$ for some positive reals b and c. Let us denoteby x and y the coordinates of P, i.e $P=(x,y)$. Accordingly,, $G=(\frac{a+b}{3},\frac{c}{3})$.

Now,

$$3|GP)^2 = 3((x-\tfrac{a+b}{3})^2+(y-\tfrac{c}{3})^2)$$

$$3|PG|^2 = |AP|^2+|BP|^2+(CP|^2-\tfrac{1}{3}(|AB|^2+|BC|^2+|AC|^2),$$

which is what we set out to prove.

Problem 5: A young man found a piece of paper on which was written a description of the position of a pirate's treasure on a desert island. The description was as follows: On the island there is a palm tree, a cedar tree and a gallows; walk from the gallows to the palm tree, counting your steps, and when you reach the palm tree turn 90º to the right, count the same number of steps and knock in a stake. Go back to the gallows, walk to the cedar counting your steps and when you reach the cedar turn 90º to the left, count the same number of steps and knock in another stake. The treasure is in the centre of the line determined by the two stakes. When he reached the island the cedar and the palm tree were there but the gallows had disappeared over the course of time. How could he find the treasure in the absence of the gallows?

Solution: Making an initial examination of the problem, the only visible items we have are the cedar and the palm tree. It therefore appears that the problem will have to be solved using these two points of reference, which we shall call C and P. Reasoning regressively, if we knew the location of the stakes the problem would be solved. Let us imagine that they have cartesian coordinates $E_1=(x_1,y_1)$ and $E_2=(x_2,y_2)$ in a fixed rectangular coordinate system. In order to determine the value of these coordinates, two pieces of information

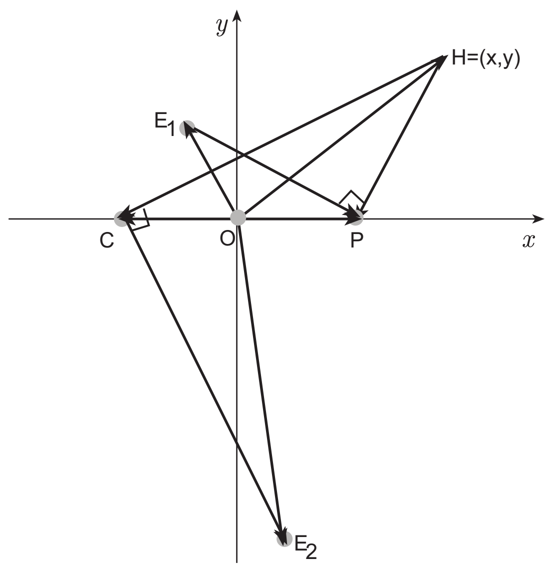

Figure 5: Cartesian representation.

are needed, but we have them: the respective 90º turns and the equal distances from the gallows to the palm tree and to the cedar tree, respectively. The coordinates of these points could be written as a function of the coordinates of the unknown point (location of the gallows) and which designate by $H = (x, y)$. Another relevant piece of information have is the direction of the turns. (*Identification, choice of elements as ordered pairs and assignment of literal coordinates to the related elements*). The most natural and simplified rectangular cartesian system appears to be the one which takes as one of its axes, for example the "x" axis, the straight line CP, and the "y" axis as the perpendicular bisector of the line segment determined by those two points (*choice of a system of coordinates*) and we assign, with no loss of generality, the cartesian coordinates $(-1; 0)$ for C and $(1; 0)$ for P. After covering the distances from the gallows to the palm tree and the cedar respectively, we can represent the points occupied by the stakes, (*representation of the figure in the system of coordinates*):

If the coordinates of the points E_1 and E_2 were determined, the solution of the problem is found. Since $\overrightarrow{HP} = \overrightarrow{OP} - \overrightarrow{OH} = (1 - x, -y)$, $\overrightarrow{HC} = -\overrightarrow{OH} + \overrightarrow{OC} = (-1 - x, -y)$ and $\overrightarrow{PE_1} = -\overrightarrow{OP} + \overrightarrow{OE_1} = (-1 + x_1, y_1)$. Besides, the scalar product of $\overrightarrow{HP}$ and $\overrightarrow{PE_1}$ is equal to zero and so $(1-x)(x_1-1) - yy_1 = 0$. Taking in account that $|HP| = |PE_1|$, one gets $(1 - x)^2 + y^2 = (x_1 - 1)^2 + y_1^2$. Solving this system of two equations, we obtain two points, one of which meets the conditions of the problem and so $E_1 = (1 - y, x - 1)$. Similarly, we get $E_2 = (y-1, -x-1)$. Accordingly, the mid point of the line segment determined by E_1 and E_2 has coordinates $(0; -1)$. To find the treasure, it is only necessary to walk from the point of the perpendicular bisector of the segment formed by the cedar and the palm tree, one unit in the negative orientation given to the y-axis.

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