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A Stage-Structure Predator-Prey Model with Functional Response

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Abstract

In this paper, a class of two species predator-prey model with functional response and harvesting, we consider a stage-structured population model with two life stages, immature and mature; and assumed the predator have different functional response between immature and mature. According to model we obtain the optimal harvesting policy and the condition for the optimal policy

Keywords: immature; mature; functional response; Hamiltonian; optimal policy

1 Introduction

The models used for resources assessment rarely take into account the total life cycle of an exploited marine population. They only consider the individuals susceptible to exploitation, which constitute the so-called stock. The exploitation stock does not contain in general larvas and old fish, because larvas and alevin are too small or absent in the potential fishing zones, and the old fish eventually leave the fishing zones, or become inaccessible to the fleet. But we notice that the fisher do not exclude the fishing of the juvenile, and that it is developing in an alarming way and without control even if there are strict measures that forbid this fishing, therefore

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to solve this problem, there must take in consideration this fishing with taking in account the juvenile stage in the system that describes the stock evolution.

2 The mathematical model

$$\begin{cases} \dot{x_0} = -\alpha x_0 - m_0 x_0 + F_1 x_1 - y\dfrac{x_0}{x_0+x_1}\varphi(x_0+x_1) \\ \dot{x_1} = \alpha x_0 - m_1 x_1 - y\dfrac{x_1}{x_0+x_1}\varphi(x_0+x_1) - q_1 E_1 x_1 \\ \dot{y} = ky\varphi(x_1+x_0) - m_2 y - q_2 E_2 y \end{cases} \tag{1}$$

where $x_0(t),\ x_1(t)$ represent the immature and mature population sizes respectively to model stage-structured population growth; $y(t)$ represent the predator population size; α represent the immature's transformation rate of mature; F_1 represent the birth rate of the immature population; $m_0,\ m_1,\ m_2$ respectively represent the death rate of the immature, the mature and the predator; $E_1,\ E_2$ respectively represent the fishing effort; $\varphi(x_1+x_0)$ represent the functional response between predator and prey. where $\alpha(t),\ m_i(t)\quad (i=0,\ 1,\ 2),\ F_1(t)$ are nonnegative function.

Now assume $p_i(i=1,\ 2)$, of the harvested resource is a fixed constant; furthermore assume that the cost, $c_i(i=1,2)$, of a unit of fishing effort is also constant, $\delta>0$ is a constant denoting the rate of discount, this objective may be expressed as maximizing:

$$\max_{E_1,\ E_2} \int_0^{\infty} e^{(-\delta t)}\{(q_1p_1x_1-c_1)E_1(t)+(q_2p_2y-c_2)E_2(t)\}dt. \tag{2}$$

where, $\quad 0\le E_1(t)\le E_1^{max}\ ,\qquad 0\le E_2(t)\le E_2^{max}\qquad \forall t\ge 0$

In order to maximize the net revenues, simultaneously can make the resources renewable, we should view the fishing problem as an optimal control problem.Our goal consists on the determination of an optimal fishing effort (E_1^*,E_2^*).

3 Application of the maximum principle

First we introduce the Hamiltonian[1]:

$$\begin{aligned} H(t,E_1,E_2,x_1,y,R,S) = {} & R[\alpha x_0 - m_1x_1 - y\frac{x_1}{x_0+x_1}\varphi(x_0+x_1) - q_1E_1x_1] \\ & +S[ky\varphi(x_1+x_0) - m_2y - q_2E_2y] \\ & +e^{(-\delta t)}\{(q_1p_1x_1-c_1)E_1(t)+(q_2p_2y-c_2)E_2(t)\}dt. \end{aligned}$$

where R or S are additional variables called the adjoint variables. If (E_1^*, E_2^*) is an optimal control and (x_1^*, y^*) is the corresponding response. The functional response is:

$$\varphi(x_0+x_1)=\begin{cases}\dfrac{b}{a}(x_0+x_1) & 0\le x_0+x_1\le a\\ b & x_0+x_1>a\end{cases}$$

the maximum principle asserts the existence of adjoint variables $R(t)$ and $S(t)$ such that the following equation are satisfied, for all t:

$$\begin{cases}\dot{R(t)} &= -\dfrac{\partial H}{\partial x_1} &= R(m_1+q_1E_1^*+y\dfrac{b}{a})-Sky\dfrac{b}{a}-e^{(-\delta t)}q_1p_1E_1^*-TF_1\\ \dot{S(t)} &= -\dfrac{\partial H}{\partial y} &= Rx_1\dfrac{b}{a}+S[m_2+q_2E_2^*-\dfrac{b}{a}k(x_0+x_1)]-e^{(-\delta t)}q_2p_2E_2^*+Tx_0\\ \dot{T(t)} &= -\dfrac{\partial H}{\partial x_0} &= T(\alpha+m_0+y\dfrac{b}{a})-R\alpha-Sky\dfrac{b}{a}\end{cases} \tag{3}$$

If we replace R by $\overline{R}e^{-\delta t}$, and S by $\overline{S}e^{-\delta t}$. the associate system (3) becomes:

$$\begin{cases}\dot{\overline{R}}=\overline{R}(m_1+\delta+q_1E_1^*+\dfrac{b}{a}y)-\overline{S}ky\dfrac{b}{a}-p_1q_1E_1^*-TF_1\\ \dot{\overline{S}}=\overline{S}[m_2+q_2E_2^*+\delta-\dfrac{b}{a}k(x_0+x_1)]+\overline{R}\dfrac{b}{a}x_1-p_2q_2E_2^*+Tx_0\\ \dot{\overline{T}}=\overline{T}(\alpha+m_0+\delta+y\dfrac{b}{a})-\overline{R}\alpha-\overline{S}ky\dfrac{b}{a}\end{cases} \tag{4}$$

The Hamiltonian become:

$$H(t,\ E_1^*,\ E_2^*,\ x_1^*,\ y^*,\ \overline{R}(t),\ \overline{S}(t))$$

$$\underset{(E_1,E_2)}{max}\left\{\begin{array}{c}=e^{-\delta t}\overline{R}[\alpha x_0-m_1x_1-q_1E_1x_1-yx_1\dfrac{b}{a}]\\ +\overline{S}e^{-\delta t}[ky\dfrac{b}{a}(x_0+x_1)-m_2-y-q_2E_2y]\\ +\overline{T}e^{-\delta t}[-\alpha x_0-m_0x_0+F_1x_1-y\dfrac{b}{a}x_0]\\ +e^{-\delta t}[(p_1q_1x_1-c_1)E_1+(p_2q_2y-c_2)E_2]\end{array}\right\}. \tag{5}$$

The $Pontryagin's$ maximum principle, for all t, (E_1^*, E_2^*) must maximize the Hamiltonian. The linearity of the Hamiltonian with respect to the controls leads to a $bang-bang$ optimal control.

$$\frac{\partial H}{\partial E_1}=e^{-\delta t}(-\overline{R}q_1x_1+p_1q_1x_1-c_1) \qquad \frac{\partial H}{\partial E_2}=e^{-\delta t}(-\overline{S}q_2y+p_2q_2y-c_2)$$

$$\begin{cases} if \quad p_1 - \dfrac{c_1}{q_1 x_1} - \overline{R}(t) > 0 & \text{then} \quad E_1^* = E_1^{max} \\ if \quad p_1 - \dfrac{c_1}{q_1 x_1} - \overline{R}(t) < 0 & \text{then} \quad E_1^* = 0 \\ if \quad p_2 - \dfrac{c_2}{q_2 y} - \overline{S}(t) > 0 & \text{then} \quad E_2^* = E_2^{max} \\ if \quad p_2 - \dfrac{c_2}{q_2 y} - \overline{S}(t) > 0 & \text{then} \quad E_2^* = 0 \end{cases} \tag{6}$$

Note that when the switching function $\overline{R}(t) - p_1 + \dfrac{c_1}{q_1 x_1}$ or $\overline{S}(t) - p_2 + \dfrac{c_2}{q_2 y}$ vanishes, the Hamiltonian becomes independent of (E_1, E_2), so the maximum principle does not specify the value of the optimal control. The most important case arises when $\overline{R}(t) - p_1 + \dfrac{c_1}{q_1 x_1}$, $\overline{S}(t) - p_2 + \dfrac{c_2}{q_2 y}$ vanishes identically over some time interval of positive length, Establishing the existence of this interval will permit us to identify the following system by deriving these two equations $\overline{R}(t) - p_1 + \dfrac{c_1}{q_1 x_1} = 0$ and $\overline{S}(t) - p_2 + \dfrac{c_2}{q_2 y} = 0$:

$$\begin{cases} p_1(m_1 + \delta + \dfrac{b}{a} y) - \dfrac{c_1}{q_1 x_1}(\delta + \dfrac{\alpha x_0}{x_1}) - (p_2 - \dfrac{c_2}{q_2 y}) k y \dfrac{b}{a} = 0 \\ p_2[m_2 + \delta - \dfrac{b}{a} k (x_0 + x_1)] - \dfrac{c_2}{q_2 y}\delta + p_1 x_1 \dfrac{b}{a} - \dfrac{c_1 b}{a q_1} = 0 \end{cases} \tag{7}$$

4 The optimal strategy

We are ready to describe definitively the optimal exploitation policy. Now we study the system (7) given in the previous section. Let us the following function denote by $\phi_1(x_1, y)$, the first equation of system (7), let $\phi_2(x_1, y)$ denote the second equation of system (7).

$$\phi_1(x_1, y) = p_1(m_1 + \delta + \frac{b}{a} y) - \frac{c_1}{q_1 x_1}(\delta + \frac{\alpha x_0}{x_1}) - (p_2 - \frac{c_2}{q_2 y}) k y \frac{b}{a} = 0 \tag{8}$$

$$\phi_2(x_1, y) = p_2[m_2 + \delta - \frac{b}{a} k (x_0 + x_1)] - \frac{c_2}{q_2 y}\delta + p_1 x_1 \frac{b}{a} - \frac{c_1 b}{a q_1} = 0 \tag{9}$$

Let $\dfrac{x_0}{x_1} = \dfrac{F}{\alpha + m_0 + \beta y}, \quad \beta = \dfrac{b}{a}, \quad m_1 + \delta = L, \quad m_0 + \alpha = K.$
From equation (8) we obtain a function $y(x_1)$:

$$y(x_1) = \frac{\dfrac{c_2 \delta \beta}{q_1 x_1} + p_2 k K \beta - \dfrac{c_2 k \beta^2}{q_2} - p_1 (K + L)\beta - \sqrt{\Delta}}{2\beta^2 (p_1 - p_2 k)}$$

where,

$$\Delta = p_1\beta^2(K-L)^2 + (\frac{c_2\beta\delta}{q_1x_1})^2 + (p_2kK\beta)^2 + (\frac{c_2k\beta^2}{q_2})^2 - \frac{2p_1\beta^2(K+L)c_2\delta}{q_1x_1}$$
$$-2p_1p_2kK\beta^2(K-L) + \frac{2p_1\beta^2k(L-K)c_2}{q_2} + \frac{2c_2\delta\beta^2kKp_2}{q_1x_1} - \frac{2c_2^2k\delta\beta^3}{q_1q_2x_1}$$
$$+\frac{2p_2k^2Kc_2\beta^3}{q_2} + \frac{4c_1\beta^2}{q_1x_1}(\delta K + \alpha F)(p_1 - p_2k)$$

From equation (9) we obtain a function $x_1(y)$:

$$x_1(y) = \frac{\frac{c_2\delta}{yq_2} + \frac{c_1\beta}{q_1} - p_2(m_2+\delta)}{p_1\beta - p_2k\beta(1 + \frac{F}{K+y\beta})}$$

The graph of function $y(x_1)$ and function $x_1(y)$ are given by $Fig.(1) and Fig.(2)$:

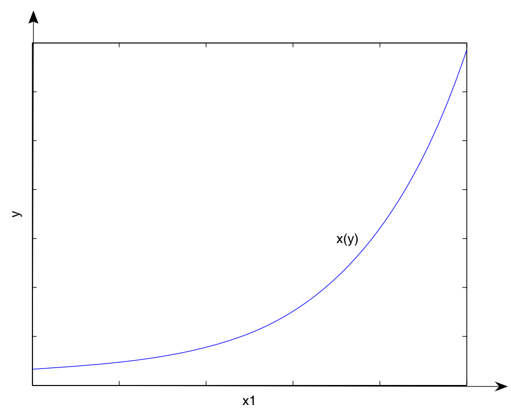

Fig.(1)

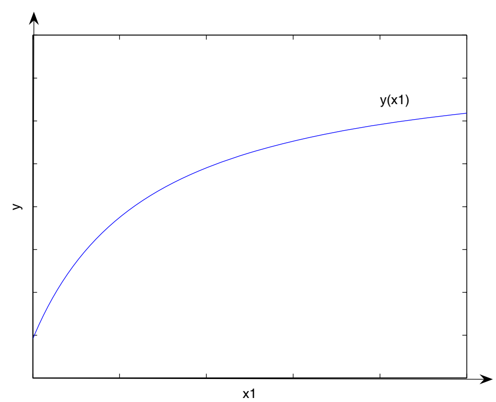

Fig.(2)

Now we describe the optimal exploitation policy,

$$\begin{cases} if & x_1 > x_1(y) & \text{then} & E_1^* = E_1^{max} \\ if & x_1 < x_1(y) & \text{then} & E_1^* = 0 \\ if & y > y(x_1) & \text{then} & E_2^* = E_2^{max} \\ if & y > y(x_1) & \text{then} & E_2^* = 0 \end{cases}$$

This optimal strategy is given and illustrate by the following $Fig.(3)$:

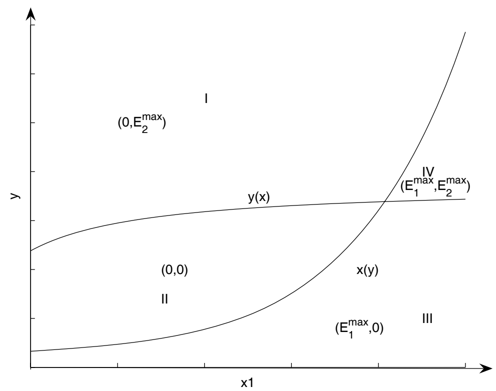

Fig.(3)

The intersect of the two curves is the equilibrium of system (1). If in the area I denote the predator biomass is large and the adult prey biomass is weak. The optimal control consists of taking $(E_1^*, E_2^*) = (0, E_2^{max})$, in order to increase the adult prey biomass as fast as possible until the trajectory reaches the area IV to take as optimal control $(E_1^*, E_2^*) = (E_1^{max}, E_2^{max})$; If in the area II denote the predator biomass and the adult prey biomass are weak, the optimal control consists of taking $(E_1^*, E_2^*) = (0, 0)$; If in the area III denote the predator biomass is weak and the adult prey biomass is large. The optimal control consists of taking $(E_1^*, E_2^*) = (E_1^{max}, 0)$; If in the area IV denote the predator biomass and the adult prey biomass are large. The optimal control consists of taking $(E_1^*, E_2^*) = (E_1^{max}, E_2^{max})$.

5 Conclusion

In the previous studies[2-4], the aim was the search of equilibrium points for a structured model and the study of the stability, but in this work, a structure predator-prey model is associated with the maximization of a total discounted net revenues derive from exploitation of the resource, and the main objective is to prove the existence of an optimal strategy for the fishing problem. By using tools of control theory, the $pontryagin's$ maximum principle, we are found the optimal strategy of the fishing problem.

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