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New Integral Inequalities for n-Time Differentiable Functions with Applications for pdfs

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Abstract

We present some new integral inequalities for $n-$time differentiable functions, defined on a finite interval. These inequalities are used to obtain some new estimations for propability density functions, as well as, inequalities involving the incomplete Beta function.

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1 Introduction

Many authors by using variants of some classical integral inequalities as for example the midpoint, trapezoid , Ostrowski, Gruess inequalities [8], [9], [10] have produced some interesting results for probability density functions of a random variable defined over a finite interval, see for example $[1]-[7]$.

In this paper we prove some new integral inequalities for $n-$time differentiable functions which in special cases lead to the following results:

Theorem 1 *Let $f\in C\left[a,b\right]$ be a function such that $f'\in L_{\infty}\left(a,b\right)$. If $f\left(x\right)\geq 0$ for all $x\in\left(a,b\right)$ and f is not identically zero on $\left(a,b\right)$, then we have the inequality*

$$\frac{1}{2}m\leq\frac{f\left(b\right)\int_a^b\left(b-t\right)f\left(t\right)dt-f\left(a\right)\int_a^b\left(t-a\right)f\left(t\right)dt}{\int_a^b\left(\left(t-a\right)^2+\left(b-t\right)^2\right)f\left(t\right)dt}\leq\frac{1}{2}M \qquad (1)$$

where $m := \inf_{x\in(a,b)} f'(x)$ *and* $M := \sup_{x\in(a,b)} f'(x)$. *If* $0 \leq a < b$, *then the above inequality* (1) *is sharp.*

Theorem 2 *Let* $f \in C[a,b]$ *be a function not identically zero on* (a,b) *and such that* $f(x) \geq 0$ *for all* $x \in [a,b]$. *Then the following inequalities hold:*

$$\frac{1}{3}\min_{x\in[a,b]} f(x) \leq \frac{\int_a^b f(x)\,dx \int_a^b (b-t)^2 f(t)\,dt}{\int_a^b \left((b-a)^3 - (t-a)^3 + (b-t)^3\right) f(t)\,dt} \leq \frac{1}{3}\max_{x\in[a,b]} f(x), \tag{2}$$

and

$$\frac{1}{3}\min_{x\in[a,b]} f(x) \leq \frac{\int_a^b f(x)\,dx \int_a^b (t-a)^2 f(t)\,dt}{\int_a^b \left((b-a)^3 + (t-a)^3 - (b-t)^3\right) f(t)\,dt} \leq \frac{1}{3}\max_{x\in[a,b]} f(x). \tag{3}$$

The inequalities (2), (3) *are sharp.*

The above inequalities (1), (2) and (3) are used to obtain some new inequalities for probability, expectation and variance of random variable defined over a finite interval as well as for the incomplete Beta function.

2 Inequalities for $n-$time differentiable functions

For our purpose we need the identity in the following Lemma:

Lemma 3 *Let* $f : [a,b] \to \mathbb{R}$ *be a* $n-$*time differentiable function in* (a,b) *such that* $f^{(n)}$ *is integrable on* $[a,b]$. *Then the following identity holds,*

$$-\frac{1}{n!}\int_a^b \int_a^b K(x,t)\, f^{(n)}(x)\, f(t)\,dxdt$$

$$= \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(f^{(k)}(a) \int_a^b (t-a)^{k+1} f(t)\,dt + f^{(k)}(b) \int_a^b (t-b)^{k+1} f(t)\,dt \right) \tag{4}$$

where the kernel $K : [a,b]^2 \to \mathbb{R}$ *is given by*

$$K(x,t) := \begin{cases} (t-x)^n & if \quad x \in [a,t] \\ -(t-x)^n & if \quad x \in (t,b] \end{cases}.$$

Proof. Let us define the following function:

$$F(x) := \int_a^x f(u)\,du, \quad x \in [a,b]. \tag{5}$$

Then, by using the integral form of remainder in Taylor's formula we have

$$\frac{1}{n!}\int_a^b\int_a^b K(x,t)\,f^{(n)}(x)\,f(t)\,dxdt=\frac{1}{n!}\int_a^b\int_a^b K(x,t)\,F^{(n+1)}(x)\,f(t)\,dxdt$$

$$=\int_a^b\left(\int_a^t\frac{(t-x)^n}{n!}F^{(n+1)}(x)\,dx+\int_b^t\frac{(t-x)^n}{n!}F^{(n+1)}(x)\,dx\right)f(t)\,dt$$

$$=\int_a^b\left(2F(t)-F(a)-F(b)-\sum_{k=1}^{n}\frac{F^{(k)}(a)(t-a)^k+F^{(k)}(b)(t-b)^k}{k!}\right)f(t)\,dt$$

which, by using (5) gives,

$$\frac{1}{n!}\int_a^b\int_a^b K(x,t)\,f^{(n)}(x)\,f(t)\,dxdt=2\int_a^b f(t)\left(\int_a^t f(x)\,dx\right)dt-\left(\int_a^b f(t)\,dt\right)^2 \quad (6)$$

$$-\sum_{k=1}^{n}\frac{f^{(k-1)}(a)\int_a^b(t-a)^k f(t)\,dt+f^{(k-1)}(b)\int_a^b(t-b)^k f(t)\,dt}{k!}. \quad (7)$$

But

$$2\int_a^b f(t)\left(\int_a^t f(u)\,du\right)dt-\left(\int_a^b f(t)\,dt\right)^2$$

$$=\int_a^b\frac{d}{dt}\left(\int_a^t f(u)\,du\right)^2dt-\left(\int_a^b f(t)\,dt\right)^2$$

$$=\left(\int_a^b f(u)\,du\right)^2-\left(\int_a^b f(t)\,dt\right)^2=0. \quad (8)$$

Using (7) in (6) we get

$$\frac{1}{n!}\int_a^b\int_a^b K(x,t)\,f^{(n)}(x)\,f(t)\,dxdt$$

$$=-\sum_{k=1}^{n}\frac{1}{k!}\left(f^{(k-1)}(a)\int_a^b(t-a)^k f(t)\,dt+f^{(k-1)}(b)\int_a^b(t-b)^k f(t)\,dt\right)$$

Finally, replacing k with $k+1$ we get the desired identity (4). ■

Theorem 4 *Assume (p,q) is a pair of conjugate exponents, that is $1\le p,q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be a $n-$time differentiable function on (a,b) such that $f^{(n)}$ is integrable on $[a,b]$. Then we have the inequalities*

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(f^{(k)}(a)\int_a^b(t-a)^{k+1}f(t)\,dt+f^{(k)}(b)\int_a^b(t-b)^{k+1}f(t)\,dt\right)\right|$$

$$\leq \begin{cases} \frac{2(b-a)^{n+\frac{2}{q}}}{n!((nq+1)(nq+2))^{\frac{1}{q}}} \left\|f^{(n)}\right\|_p \|f\|_p & if \quad f^{(n)} \in L_p[a,b], \ 1<p<\infty; \\ \frac{2(b-a)^{n+2}}{(n+2)!} \left\|f^{(n)}\right\|_\infty \|f\|_\infty & if \quad f^{(n)} \in L_\infty[a,b]; \\ \frac{(b-a)^n}{n!} \left\|f^{(n)}\right\|_1 \|f\|_1 & if \quad f^{(n)} \in L_1[a,b]. \end{cases}$$

Proof. Using Hölder's inequality we have

$$\frac{1}{n!}\left|\int_a^b\int_a^b K(x,t)\, f^{(n)}(x)\, f(t)\, dxdt\right|$$

$$\leq \frac{1}{n!}\left(\int_a^b\int_a^b |K(x,t)|^q\, dxdt\right)^{\frac{1}{q}}\left(\int_a^b\int_a^b \left|f^{(n)}(x)\, f(t)\right|^p dxdt\right)^{\frac{1}{p}}$$

$$= \frac{1}{n!}\left(\int_a^b\left(\int_a^t (t-x)^{nq}\, dx + \int_t^b (x-t)^{nq}\, dx\right) dt\right)^{\frac{1}{q}}$$
$$\times\left(\int_a^b \left|f^{(n)}(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}}$$

$$= \frac{1}{n!}\frac{2^{\frac{1}{q}}(b-a)^{n+\frac{2}{q}}}{((nq+1)(nq+2))^{\frac{1}{q}}}\left\|f^{(n)}\right\|_p \|f\|_p. \tag{9}$$

Combining (8) with (4) we get the first conclusion.

We also have,

$$\frac{1}{n!}\left|\int_a^b\int_a^b K(x,t)\, f^{(n)}(x)\, f(t)\, dxdt\right|$$

$$\leq \frac{1}{n!}\int_a^b\left(\int_a^b |K(x,t)\, f(t)|\, dt\right)\left|f^{(n)}(x)\right| dx$$

$$\leq \frac{1}{n!}\left\|f^{(n)}\right\|_\infty \|f\|_\infty \int_a^b\int_a^b |K(x,t)|\, dxdt$$

$$= \frac{1}{(n+1)!}\left\|f^{(n)}\right\|_\infty \|f\|_\infty \int_a^b \left((b-t)^{n+1} + (t-a)^{n+1}\right) dt$$

$$= \frac{2(b-a)^{n+2}}{(n+2)!}\left\|f^{(n)}\right\|_\infty \|f\|_\infty. \tag{10}$$

Combining the identity (4) in Lemma 3 with (9) we get the second conclusion.

Finally, we have

$$\frac{1}{n!}\left|\int_a^b\int_a^b K(x,t)\,f^{(n)}(x)\,f(t)\,dxdt\right|$$

$$\leq \frac{1}{n!}\int_a^b\int_a^b |K(x,t)|\left|f^{(n)}(x)\,f(t)\right|dxdt$$

$$\leq \frac{1}{n!}\max_{(x,t)\in[a,b]\times[a,b]}|K(x,t)|\int_a^b\int_a^b\left|f^{(n)}(x)\,f(t)\right|dxdt$$

$$= \frac{(b-a)^n}{n!}\|f\|_1\left\|f^{(n)}\right\|_1. \tag{11}$$

Combining (10) with (4) we get the last conclusion. ■

In some cases, the second inequality in Theorem 4 can be replaced by a better one, which will be used in the next section to prove the Theorems 1 and 2:

Theorem 5 *Let $f:[a,b]\to\mathbb{R}$ be a $n-$time differentiable function on (a,b) such that $f^{(n)}$ is integrable and bounded on $[a,b]$. Assume that f is not identically zero on (a,b) and $f\geq 0$ on (a,b). If n is an odd non negative integer, then the following inequality holds:*

$$\frac{1}{(n+1)!}m$$

$$\leq -\frac{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(f^{(k)}(a)\int_a^b(t-a)^{k+1}f(t)\,dt+f^{(k)}(b)\int_a^b(t-b)^{k+1}f(t)\,dt\right)}{\int_a^b\left((t-a)^{n+1}+(b-t)^{n+1}\right)f(t)\,dt}$$

$$\leq \frac{1}{(n+1)!}M. \tag{12}$$

where

$$m:=\inf_{x\in(a,b)}f^{(n)}(x), \qquad M:=\sup_{x\in(a,b)}f^{(n)}(x).$$

If $0\leq a<b$, then inequality (11) is the best possible.

Proof. Since n is odd, we have

$$K(x,t)=|t-x|^n$$

Therefore, for all $t,x\in(a,b)$ we get

$$\frac{|t-x|^n}{n!}m\leq\frac{K(x,t)}{n!}f^{(n)}(x)\leq\frac{|t-x|^n}{n!}M.$$

Integrating the above estimation with respect to x from a to b we get,

$$\frac{(t-a)^{n+1}+(b-t)^{n+1}}{(n+1)!}m \leq \frac{1}{n!}\int_a^b K(x,t)\, f^{(n)}(x)\, dx$$
$$\leq M\frac{(t-a)^{n+1}+(b-t)^{n+1}}{(n+1)!}.$$

Now, by multiplying the last estimation with $f(t) \geq 0$ on $[a,b]$ and integrating the resulting inequality with respect to t from a to b, we obtain the following estimation:

$$\frac{m}{(n+1)!}\int_a^b \left((t-a)^{n+1}+(b-t)^{n+1}\right) f(t)\, dt$$
$$\leq \frac{1}{n!}\int_a^b\int_a^b K(x,t)\, f^{(n)}(x)\, f(t)\, dxdt$$
$$\leq \frac{M}{(n+1)!}\int_a^b \left((t-a)^{n+1}+(b-t)^{n+1}\right) f(t)\, dt. \quad (13)$$

Since f is continuous, non negative and not identically zero on (a,b), we clearly have

$$\int_a^b \left((t-a)^{n+1}+(b-t)^{n+1}\right) f(t)\, dt > 0. \quad (14)$$

Finally, using the identity (4) in (12) and afterwards dividing the resulting estimation with (13) we get the desired inequality (11). Now, let us consider the case $0 \leq a < b$ and let us choose $f(x) = x^n,\ x \in [a,b]$ Clearly, f satisfies the required assumptions of this Theorem and an easy calculation yields, that the equality in (11) holds. ■

Corollary 6 *Let n be an odd positive integer. Let $f \in C^{2n+2}[a,b]$. If $f^{(n+2)}(x) \geq 0$ for all $x \in (a,b)$ and $f^{(n+2)}$ is not identically zero on (a,b), then there exists one $\xi \in [a,b]$ such that*

$$\frac{\sum_{i=0}^{n-1}\frac{(-1)^i}{(i+1)!}\left(f^{(i+n+2)}(b)\, R_{i+1}\left(f^{(n-i)};a,b\right) - f^{(i+n+2)}(a)\, R_{i+1}\left(f^{(n-i)};b,a\right)\right)}{R_{n+1}(f;a,b)-R_{n+1}(f;b,a)}$$
$$= f^{(2n+2)}(\xi),$$

where by

$$R_n(f;a,b) := f(b) - \sum_{k=0}^{n}\frac{(b-a)^k}{k!}f^{(k)}(a) = \frac{1}{n!}\int_a^b (b-x)^n f^{(n+1)}(x)\, dx$$

is denoted the remainder in Taylor's formula.

Proof. Applying Theorem 5 for the continuous on $[a,b]$ mapping $f^{(n+2)}$, and using the integral form of Taylor's remainder we readily get

$$m \leq \frac{\sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \left(f^{(k+n+2)}(b) \, R_{k+1}\left(f^{(n-k)}; a, b\right) - f^{(k+n+2)}(a) \, R_{k+1}\left(f^{(n-k)}; b, a\right)\right)}{R_{n+1}(f; a, b) - R_{n+1}(f; b, a)} \tag{15}$$

$$\leq M, \tag{16}$$

where $m := \min_{x\in[a,b]} f^{(2n+2)}(x)$ and $M := \max_{x\in[a,b]} f^{(n)}(x)$.

Since $f^{(n+2)}$ is continuous on $[a,b]$, according to the intermediate value theorem, from (14) we immediately get the conclusion. ■

3 The proofs of Theorems 1 and 2

Proof. [Proof of Theorem 1] Applying the inequality (11) in Theorem 5 for $n = 1$, we immediately get the desired result. ■

Proof. [Proof of Theorem 2] We define the function $F(x) := \int_a^x f(t)\,dt, \quad x \in [a,b]$. According to the assumptions for f we clearly have that F satisfies the conditions of lemma 1. Now, if we write inequality (1) for $F,$ we get

$$\frac{m}{2} \leq \frac{\int_a^b f(x)\,dx \int_a^b (b-t)\left(\int_a^t f(x)\,dx\right)dt}{\int_a^b \left((t-a)^2 + (b-t)^2\right)\left(\int_a^t f(x)\,dx\right)dt} \leq \frac{M}{2} \tag{17}$$

where $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$.

Moreover, using the integration by parts we readily get

$$\int_a^b (b-t)\left(\int_a^t f(x)\,dx\right)dt = \frac{1}{2}\int_a^b (b-t)^2 f(t)\,dt \tag{18}$$

and

$$\int_a^b \left((t-a)^2 + (b-t)^2\right)\left(\int_a^t f(x)\,dx\right)dt$$
$$= \frac{1}{3}\int_a^b \left((b-a)^3 - (t-a)^3 + (b-t)^3\right) f(t)\,dt. \tag{19}$$

Putting (16) and (17) in (15) we get the first conclusion (2) of Theorem 2.

Now, if we write inequality (1) for the function $F(x) := \int_x^b f(t)\,dt$, $x \in [a,b]$, we get

$$\frac{m}{2} \leq \frac{-\int_a^b f(x)\,dx \int_a^b (t-a)\left(\int_t^b f(x)\,dx\right)dt}{\int_a^b \left((t-a)^2 + (b-t)^2\right)\left(\int_t^b f(x)\,dx\right)dt} \leq \frac{M}{2}$$

and working similarly as above, we get the second conclusion (3). Finally, choosing $f(x) = 1$ in (2), (3) we see that the equalities hold. ■

At the next section 4, we apply the Theorems 1 and 2 for some applications for probability density functions (pdfs).

4 Applications for pdfs

Using the inequality (1) in Theorem 1, we immediately get the following proposition:

Proposition 7 *Let X be a random variable having the probability density function $f : [a,b] \to \mathbb{R}$ Assume that $f \in C[a,b]$ and $f' \in L_\infty(a,b)$. Then the following inequality holds:*

$$\frac{1}{2}\inf_{x\in(a,b)} f'(x) \le \frac{f(b)(b-\mu) - f(a)(\mu-a)}{\sigma^2+\mu^2-2(a+b)\mu+a^2+b^2} \le \frac{1}{2}\sup_{x\in(a,b)} f'(x),$$

where μ and σ^2 are respectively the expectation and the variance of the random variable X.

Proposition 8 *Let X be a random variable having the probability density function $f : [a,b] \to \mathbb{R}$ Assume that $f \in C[a,b]$. Then for all $x \in [a,b]$ the following inequalities hold:*

$$\frac{1}{3}\min_{t\in[a,x]} f(t) \le \frac{\Pr(X\le x)\, E_x\left((X-a)^2\right)}{(b-a)^3 \Pr(X\le x) + E_x\left((X-a)^3\right) + E_x\left((X-b)^3\right)} \le \frac{1}{3}\max_{t\in[a,x]} f(t) \tag{20}$$

and

$$\frac{1}{3}\min_{t\in[a,x]} f(t) \le \frac{\Pr(X\le x)\, E_x\left((X-b)^2\right)}{(b-a)^3 \Pr(X\le x) - E_x\left((X-a)^3\right) - E_x\left((X-b)^3\right)} \le \frac{1}{3}\max_{t\in[a,x]} f(t) \tag{21}$$

where

$$\Pr(X\le x) := \int_a^x f(t)\,dt, \quad x\in[a,b]$$

is the cumulative distribution function of X, and

$$E_x(g(X)) := \int_a^x g(u) f(u)\,du, \qquad x\in[a,b].$$

is the incomplete expectation of the random variable $g(X)$

Proof. If we choose $b = x$ in (2) and (3), we respectively get the desired inequalities. ■

Now, let us consider the *Beta function*

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad p,q > -1$$

and the *incomplete Beta function*

$$B(x;p,q) := \int_0^x t^{p-1}(1-t)^{q-1}\,dt.$$

We consider the random variable X having the pdf

$$f(t) := \frac{t^{p-1}(1-t)^{q-1}}{B(p,q)}, \quad t \in [0,1]. \tag{22}$$

Clearly, for all $n \in \mathbb{N}$ we have

$$E_x(X^n) = \frac{B(x;p+n,q)}{B(p,q)} \tag{23}$$

and

$$E_x((1-X)^n) = \frac{B(x;p,q+n)}{B(p,q)}. \tag{24}$$

Further, we have (see [1])

$$\min_{x\in[0,1]} f(x) = 0 \text{ and } \max_{x\in[0,1]} f(x) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}B(p,q)}. \tag{25}$$

Now, applying for f, as defined by (20), the inequalities (18), (19) in Proposition 8, and using (21),(22),(23) we readily get the following Proposition.

Proposition 9 *Let X be a Beta random variable with the parameters (p,q), $p,q > 1$. Then we have the inequalities*

$$0 \le \frac{B(x;p,q)\,B(x;p+2,q)}{B(x;p,q)+B(x;p+3,q)-B(x;p,q+3)} \le \frac{1}{3}\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}},$$

$$0 \le \frac{B(x;p,q)\,B(x;p,q+2)}{B(x;p,q)-B(x;p+3,q)+B(x;p,q+3)} \le \frac{1}{3}\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}$$

for all $x \in [0,1]$.

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