

Deriving Operational Equations

G. L. Silver

Los Alamos National Laboratory*
P.O. Box 1663, MS E517
Los Alamos, NM 87545, USA
gsilver@lanl.gov

*Los Alamos National Laboratory is operated by the Los Alamos National Security, LLC for the National Nuclear Security Administration of the U.S. Department of Energy under Contract No. DE-AC52-06NA25396.

Abstract

The four-point rectangular data array has traditionally been represented by the bilinear equation. A quadratic equation for the design can be developed with or without the shifting operator. The method involving the operator is a general approach to the problem of generating equations for geometric data arrays. It is easy to apply and it is a suggestive instrument as illustrated by examples.

1. Introduction

The bilinear equation represents the traditional method for interpolating four data in rectangular array. It is versatile and easy to apply but it does not estimate curvature and it does not apply to the diamond configuration. Alternatives to the bilinear equation are provided by the shifting operator, $\exp(hx)F(x)=F(x+h)$. It has recently been applied to derive interpolating equations for data in various arrays such as four- and five-point rectangles and diamonds, and cubes [1-3]. The operator-derived equations have the advantage that they estimate curvature. They can be better representations of surfaces than the representations rendered by the bilinear and trilinear equations. This judgment is based on comparisons of sums of squares of deviations. Data in geometric designs make useful illustrations of applied operational calculus.

The new approach turns on applying operators disguised in trigonometric identities [4,5]. Identities in one parameter are converted into relationships among points on a curve. Identities in two parameters are converted into previously unperceived relationships among points on surfaces. Identities that are homogeneous in sine terms were the first to be applied to these ends but the homogeneity requirement is not absolute [4,6]. Interpolating equations so obtained offer the advantage of economy of experimental effort: curvature can be detected by means of a few measurements. That is an important consideration in view of the high costs of laboratory work [1-3].

This paper illustrates that some of the equations rendered by the shifting operator can also be obtained by conventional methods. There is no clear boundary between the category of interpolating equations requiring the shifting operator for their development and the category of equations that do not depend on the operator. It is a subject that remains to be explored. These remarks are made clearer by examples. They illustrate that there is room for further contributions to operational calculus as applied to geometry.

2. The four-point rectangle, first polynomial method

A diagram of a nine-point rectangle appears in Fig. 1. Methods for deriving operational interpolating equations for the six-point array ABCFED and the nine-point array ACIG have been illustrated in Refs. [7,8]. In order to apply these methods, equations for the four-point subspaces are needed. Ref. [4] illustrates the derivation of equations for the center and side points of the rectangle in Fig. 1. They are Eqs. (1) and (2), respectively.

$$E^2 = (GI - AC)(CI - AG) / [(I - A)^2 - (C - G)^2] \quad (1)$$

$$D^2 = (A + G)^2(CI - AG) / [(I + C)^2 - (A + G)^2] \quad (2)$$

Let each letter on both sides of Eq. (1) be augmented by adding a term denoted T. Take the square root of the augmented expressions, subtract T from them, and find the limits of the differences as T approaches infinity. The result is Eq. (3). A similar procedure applied to Eq. (2) yields Eq. (4). These are the new center and side point formulas, respectively, in Fig. 1. They are exact on bilinear numbers and their squares. Rotate Fig. 1 to obtain analogous formulas for side points B, F, and H.

$$E = \frac{[(C^2 + G^2)(I + A) - (A^2 + I^2)(C + G) + 2(CI(A - G) + GA(I - C))]}{[2((C - G)^2 - (I - A)^2)]} \quad (3)$$

$$D = [3(A^2 + G^2) + C^2 + I^2 + 2A(5G - 2C - 2I) - 2CI - 4G(I + C)] / 8(A + G - C - I) \tag{4}$$

Pearson's equation for the nine-point rectangle is Eq. (5) [9]. R represents an interpolated number at specified (x, y) coordinates in the -1 .. 1 coordinate system.

$$R = E + (F - D)x/2 + (H - B)y/2 + (A - C - G + I)xy/4 + (F - 2E + D)x^2/2 + (H - 2E + B)y^2/2 + [A + C + G + I + 4E - 2(F + D + B + H)]x^2y^2/4 \tag{5}$$

Let Eqs. (3) and (4), and the analogous expressions for side points B, F, and H, be substituted into Eq. (5). The result simplifies to Eq. (6), the operational interpolating equation for the four-point rectangle ACIG. It is exact on bilinear numbers and their squares. Eq. (6) was first obtained by the described method. Eq. (6) seems to be dependent on the shifting operator for its derivation but this conclusion is misleading.

$$R = (A + C + G + I)/4 - x2c - y2c + (I + C - A - G)x/4 + (G + I - A - C)y/4 + (I + A - C - G)xy/4 + (x2c)x^2 + (y2c)y^2 \tag{6}$$

$$x2c = (I + A - C - G)(I + C - G - A) / (8(G + I - A - C)) \tag{7}$$

$$y2c = (I + A - C - G)(G + I - A - C) / (8(I + C - A - G)) \tag{8}$$

3. The four-point rectangle, second polynomial method

The bilinear equation for the four-point rectangle ACIG is Eq. (9). The result of squaring an arbitrary bilinear expression is illustrated by Eq. (10).

$$R = (A + C + G + I)/4 + (I + C - A - G)x/4 + (G + I - A - C)y/4 + (I + A - C - G)xy/4 \tag{9}$$

$$R = (k + px + qy)^2 = k^2 + 2pkx + 2kqy + p^2x^2 + 2pqxy + q^2y^2 \tag{10}$$

Eq. (10) contains three unknowns, k, p, and q. They appear in the coefficients of the linear and the cross-product terms on the right-hand side of Eq. (10). The bilinear equation, Eq. (9), also contains linear and cross-product terms. In order to express k, p, and q in terms of A, C, G, and I, form three equations as illustrated by Eqs. (11)-(13).

$$2kp - (I + C - A - G)/4 = 0 \tag{11}$$

$$2kq - (G + I - A - C)/4 = 0 \tag{12}$$

$$2pq - (I + A - C - G)/4 = 0 \quad (13)$$

Let Eqs. (11)-(13) be solved simultaneously. Two sets of solutions result from this operation. To obtain p^2 , q^2 , and k^2 , such as are present in Eq. (10), square p , q , and k in each set. This operation yields identical expressions for the corresponding p^2 , q^2 , and k^2 . The denominator of k^2 vanishes when it is substituted with bilinear numbers like [1,3,7,9] as [A,C,G,I], respectively. This implies that k^2 is not an acceptable estimate of the center of four data in rectangular array. However, p^2 and q^2 estimate the two quadratic-term coefficients in Eq. (10) so let them be denoted by $x2c$ and $y2c$ as in Eqs. (14) and (15), respectively. An additional relationship is needed to estimate the center point. That relationship is Eq. (16). It is determined from the general form of the quadratic equation, the data, and their coordinates, as illustrated in the Appendix.

$$x2c = (I + A - C - G)(G + A - I - C) / [8(A + C - G - I)] \quad (14)$$

$$y2c = (I + A - C - G)(A + C - G - I) / [8(G + A - I - C)] \quad (15)$$

$$\text{new constant term in Eq. (9)} = (A + C + G + I)/4 - x2c - y2c \quad (16)$$

The right-hand side of Eq. (9) can be completed with the new expression for the constant term and then adding the quadratic-terms $(x2c)x^2$ and $(y2c)y^2$ as in Eq. (10). The result is an interpolating equation that is exact on bilinear numbers and their squares. It is identical to Eq. (6), the equation found with the aid of the shifting operator. The second method is simple and straightforward. It illustrates that the quadratic equation for the four-point rectangle can be obtained with or without assistance of the shifting operator. A similar approach develops the quadratic equation for the eight-point cube [3].

4. The five-point rectangle

The identity in Eq. (17) is homogeneous in sine terms. Multiply the identity by $F(x,y)^4$, the fourth power of the unknown function at the center point, E, in Fig. 1 [4,5]. The result is an expression relating E to four other vertices in the figure, Eq. (18). It is exact on linear numbers when used as exponents of a common base as in 2^x .

$$\sin(x + y)^2 + \sin(x - y)^2 + 2\sin(x + y)\sin(x - y)(2\cos(x)^2 - 1) = (2\sin(x)\cos(x))^2 \quad (17)$$

$$E^2 = \frac{[(F - D)^2(F + D)^2 - (I - A)(C - G)(F + D)^2]}{[(I - A)^2 + (C - G)^2 - 2(I - A)(C - G)]} \quad (18)$$

Remove F from Eq. (18) by means of the substitution $F = (IC - AG + D^2)^{(1/2)}$ and thereby obtain an expression for side point D in terms of A, C, E, G, and I [4]. The expression is simplified by squaring it. Even in the squared form it is complicated so it is stored in the memory of the computer. Add a parameter T to each term in the stored expression, take the square root of the result, and then subtract T from it. The limit of the difference as T approaches infinity is Eq. (19).

$$D = (AC + IG - 2AG)(C + I - A - G) - E(I + G - C - A)^2 / (4(IC + AG - AC - IG)) \tag{19}$$

Fig. 1 can be rotated so that formulas analogous to Eq. (19) are obtained for vertices B, F, and H. The four expressions are substituted into Pearson's equation, Eq. (5) above. The result is complicated but it simplifies to Eq. (20), a polynomial equation. That equation is potentially suitable for interpolating the five-point rectangle. It is exact on bilinear numbers and their squares but it is not useful if A=I or G=C. Another five-point polynomial equation can then be selected [10].

$$R = E + (I + C - A - G)x/4 + (G + I - C - A)y/4 + (I + A - C - G)xy/4 + (G - I - C + A)(E(G + A - C - I) + IC - AG)x^2 / (4(I - A)(G - C)) - (G + I - C - A)(E(G + I - A - C) + AC - IG)y^2 / (4(I - A)(G - C)) \tag{20}$$

The center point expression in Eq. (3) can replace the letter E in Eq. (20). Simplification yields Eq. (6), an equation for the four-point rectangle. This represents a third method for generating the four-point, polynomial equation.

5. Alternative representation of sine and cosine

The preceding sections have illustrated the derivation of polynomial-type equations for data in a rectangular array. The sine and cosine functions can be applied in another form in order to generate an exponential-type interpolating equation for the four-point rectangular array. The same equation can be obtained by standard methodology so the shifting operator is presently unnecessary. The operator approach is tedious by comparison but part of it is useful as an illustration of how the operator can be applied.

In Eqs. (21)-(24) the italic letter *I* represents the square root of (-1). Note that the hyperbolic functions could be used instead of the circular functions.

$$2\cos(M + N) + 2\cos(M - N) = JK + 1/(JK) + J/K + K/J \tag{21}$$

$$2\cos(M + N) - 2\cos(M - N) = JK + 1/(JK) - J/K - K/J \tag{22}$$

$$(2I)\sin(M + N) + (2I)\sin(M - N) = JK - 1/(JK) + J/K - K/J \quad (23)$$

$$(2I)\sin(M + N) - (2I)\sin(M - N) = JK - 1/(JK) - J/K + K/J \quad (24)$$

The left-hand sides of Eqs. (21)-(24) can be multiplied by unity in the form of $F(x,y)/F(x,y)$ or (E/E) . They are thereby converted to $(I + A + C + G)/E$, $(I + A - C - G)/E$, $(I - A + C - G)/E$, $(I - A - C + G)/E$, respectively [4,6]. If the data at vertices A, C, E, G, I in Fig. 1 are known, then Eqs. (21)-(24) contain only J and K as the unknowns on their right-hand sides. This circumstance permits the introduction of two new unknowns, Q and T, into the left-hand sides of Eqs. (21)-(24). We now have four equations in four unknowns. Those equations can be rewritten as Eqs. (25)-(28) and they are amenable to solution as a simultaneous set. The interested reader can pursue the details in Ref. [6].

$$[T(I + A + C + G) + 4Q] / (E + Q) = JK + 1/(JK) + J/K + K/J \quad (25)$$

$$[T(I + A - C - G)] / (E + Q) = JK + 1/(JK) - J/K - K/J \quad (26)$$

$$[T(I - A + C - G)] / (E + Q) = JK - 1/(JK) + J/K - K/J \quad (27)$$

$$[T(I - A - C + G)] / (E + Q) = JK - 1/(JK) - J/K + K/J \quad (28)$$

The exponential-type equation for the four-point rectangle, developed as indicated by Eqs. (25)-(28), is Eq. (29) [6]. In the equation, J is the square root of $(C - I)/(A - G)$ and K is the square root of $(G - I)/(A - C)$. Two things about Eq. (29) are noteworthy.

$$R = [J^{(x+1)}K^{(y+1)}(A - G)(A - C) + AI - CG] / (A - C - G + I) \quad (29)$$

(1) Eq. (29) was originally developed by means of the shifting operator [6]. The development is tedious and unnecessary. We may recognize that an exponential-type equation for the four-point rectangle can take the form of Eq. (30).

$$R = (P)J^{(x+1)}K^{(y+1)} + S \quad (30)$$

Four subsidiary equations can be developed from Eq. (30) by taking the data A, C, G, I as in Fig. 1 and recognizing that their coordinates are $(-1,-1)$, $(1,-1)$, $(-1,1)$, and $(1,1)$, respectively. The four subsidiary equations can be solved as a simultaneous set. When the solutions for J, K, P, and S are obtained they can be substituted into Eq. (30). The result is Eq. (29), the same equation obtained by the roundabout method of applying the shifting operator to the circular functions as in Eqs. (21)-(28).

The point of these remarks is that we can pay too much attention to the shifting operator. The cost of this preoccupation is the failure to recognize commonplace things. The same point is implicit in section 3 above. The shifting operator is a legitimate way to generate interpolating equations for data in geometric arrays but it is not the only method for that purpose. It is reassuring to know that the shifting operator yields the same results that are obtained by standard methodology. On the other hand, that observation raises an interesting historical question: Why have the bilinear and trilinear equations monopolized the representations of four and eight data in rectangular and cubical arrays, respectively?

Implicit in the preceding discussion is an interesting observation. There are apparently two groups of interpolating equations for data in geometric arrays: those equations whose derivations depend on the shifting operator and those equations that can be derived without the operator. The boundary separating the two groups is not clear.

(2) Functions like 2^x are monotonic-increasing with x . The slopes of such functions never change their signs. What happens when experimental data imply a change of sign in slope? For example, suppose $A=70$, $C=180$, $G=90$, $I=75$. Eq. (29) then yields Eq. (31).

$$R = (-17.6)(2.291I)^{(x+1)}(0.369I)^{(y+1)} + 87.6 \quad (31)$$

Eq. (31) can be plotted in terms of its real parts. That is a common way of using equations containing imaginary or complex numbers. The difficulty with this method is that it often generates surfaces with extrema on their boundaries. Ordinarily, there is no justification for boundary extrema when a surface is generated from only four data in rectangular array. When only four data are available, extrema are usually artifacts of the interpolating equations. This remark applies not only to Eq. (31) but to all four-point operational equations. Boundary extrema are signals to decline the equations in favor of something else. From a practical point of view, the problem of unjustified extrema can sometimes be ameliorated by the following artifice.

.... If the imaginary number occurs under the exponent containing x , change it to a real, positive number and multiply the leading coefficient in the equation by $(-x)$.

.... If the imaginary number occurs under the exponent containing y , change it to a real, positive number and multiply the leading coefficient in the equation by $(-y)$.

.... If the imaginary number occurs under both exponents, change them both to real, positive numbers and multiply the leading coefficient in the equation by (xy) .

Eq. (31) thus becomes Eq. (32). The new equation does not generate boundary extrema. This artifice does not always eliminate boundary extrema but it illustrates the idea that more imaginative approaches are needed. There is room for improvements when

four data in rectangular array are to be represented by exponential-type equations. The usefulness of equations like Eq. (32) remains to be determined in the laboratory.

$$R = (-17.6)(xy)(2.291)^{(x+1)}(0.369)^{(y+1)} + 87.6 \quad (32)$$

7. Discussion

It is a curiosity that identities in the circular functions have been exclusively regarded as relationships among angles. When treated by the shifting operator they are transformed into relationships among points in space. This approach is a simple and versatile way to generate interpolating equations for data in geometric arrays. The equations present themselves in many ways. Verifying their properties makes interesting demonstrations of the shifting operator and its applications [6,10-14].

The bilinear equation generates center and side point expressions that are functionally related. That is, the determinant of the Jacobian of the bilinear expressions for B, D, E, F in Fig. 1 is zero. The determinant is nonzero when applied to the operational expressions for the same points.

The shifting operator can address many problems besides the interpolation of data in geometric arrays. At present, we do not have a good idea of its limitations so we have to be careful. This remark is illustrated by a putative expression for the “halfth” derivative as rendered by the operator. It appears as Eq. (33) applied to the evaluation of the “halfth” derivative of the function $u(x)=x^2$ at $x=4$.

$$(u(x))^{(1/2)}(d(u(x))/dx)^{(1/2)} = (16)^{(1/2)}(8)^{(1/2)} \approx 11.3 \quad (33)$$

Eq. (33) seems to have some appealing properties. For example, the zeroth derivative of x^2 at $x = 4$ is 16, the first derivative is 8, and the second derivative is 2. The derivative-order sequence is [0, 1/2, 1, 2] and the derivative-value sequence is [16, 11.3, 8, 2]. Both orders are monotonic. According to Eq. (33), the derivative of a constant is zero. These illustrations accord with intuition. On the other hand, when the function is monotonic-decreasing Eq. (33) says that its “halfth” derivative is an imaginary number. The surprise is an artifact of the square-root operator. The question is this: If the shifting operator is to be useful in the fractional calculus, should it produce something better than Eq. (33)? The operator methods are potentially useful instruments but they are not always necessary and they may occasionally mislead us.

References

- [1] G. L. Silver, Analysis of four-point grids: the diamond configuration. *Appl. Math. Comput.* 131 (2002) 215-221.
- [2] G. L. Silver, Analysis of five-point grids: the diamond configuration. *Appl. Math. Comput.* 144 (2005) 389-395.
- [3] G. L. Silver, Analysis of three-dimensional grids: the eight-point cube. *Appl. Math. Comput.* 153 (2004) 467-473.
- [4] G. L. Silver, Operational method of data treatment. *J. Comput. Chem.* 6 (1985) 229-236.
- [5] G. L. Silver, Analysis of three-dimensional grids: Sine-and-cosine equation for the nine-point prismatic array. *Appl. Math. Comput.* 183 (2006) 617-621.
- [6] G. L. Silver, Applications of the operational method. USDOE report MLM-3670 (November 1990).
- [7] G. L. Silver, The method of roots and new Latin square equations. *Qual. Eng.* 10 (1998) 625-631.
- [8] G. L. Silver, Center points and grid equations from sub-grid equations. *Appl. Math. Comput.* 189 (2007) 1467-1475.
- [9] G. L. Silver, Latin squares, central tendencies, and cubes. *Qual. Eng.* 9 (1996-97) 129-133. Pearson's equation is Eq. (1) in the citation. It is taken from Ref. [5] therein but Pearson did not make it clear in his manuscript.
- [10] G. L. Silver, Analysis of three-dimensional grids: alternative methods for five- and nine-point cubes. *Appl. Math. Comput.* 176 (2006) 37-43.
- [11] G. L. Silver, Analysis of three-dimensional grids: Interpolating the nine-point prism. The shifting operator as an analytical instrument. *Appl. Math. Comput.* 186 (2007) 1694-1700.
- [12] G. L. Silver, Operational methods for data interpolation. USDOE report MLM-3277 (August 1985).

[13] G. L. Silver, Extensions of the operational method. USDOE report MLM-3382 (August 1986).

[14] G. L. Silver, Operational interpolation of data in triangular array. USDOE report MLM-3689 (May 1991). See pp. 20-34.

Appendix

The form of a quadratic equation for the four-point rectangle is Eq. (34). The coordinates and the four data are $(-1,-1,A)$, $(1,-1,C)$, $(-1,1,G)$ and $(1,1,I)$. See Fig. 1. Form four simultaneous equations from the data and Eq. (34) and solve them for the coefficients k , xc , yc , xy . This yields Eq. (16) for the constant term in Eq. (6).

$$R = k + (xc)x + (yc)y + (xyc)xy + (x2c)x^2 + (y2c)y^2 \quad (34)$$

G	H	I
D	E	F
A	B	C

Fig. 1. The nine-point rectangle.

Received: July 26, 2007