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The Stability and on Essential Components of the Solution Set of Generalized Quasi-Variational Inequalities[1]

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Abstract

In this work, we consider firstly the existence of the solution of generalized quasi-variational inequalities;secondly,we discuss the stability and on essential components of it's solution set.

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1 Introduction

In 1985,With the method of generalized KyFan—minimax inequality[1], Tan.K.K[4] discussed the existence of the solution of generalized quasi-variational inequality problem in the local convex Hausdorff topological space. In 1990, Ding.X.P and Tan.K.K[9] proved the two theorems of generalized quasi-variational inequality's solution set by the Himmlbesg fixed point, which improved the

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same conclusion of shi-Tan[4].In this paper ,we discuss the stability and connected essential components of the solutions set of generalized quasi-variational inequality problem(for short GQVI), which improve the corresponding result[7].

2 Preliminaries

Definition 2.1 1) a set—valued mapping $F : Y \to 2^X$is called upper semi-continuity in $y \in Y$, if O is any open set in X, and $F(y) \subset O$,there exists the open neighborhood $U(y)$ of y,such that $F(y^{'}) \subset O$ for every $y^{'} \subset U(y)$. 2) $F : Y \to 2^X$ is said to be lower semi-continuity in $y \in Y$,if O is any open set in X and $F(y) \bigcap O \neq \phi$,there exists the open neighborhood $U(y)$ of y,such that $F(y^{'}) \bigcap O \neq \phi$,for every $y^{'} \subset U(y)$. 3) F is to be continuity in $y \in Y$,if F is both upper and lower semi-continuity in y. 4) F is called continuity in Y,if F is continuity for every $y \in Y$. 5) F is a usco(upper semi-continuity)mapping,if F is semi-continuity in Y and $F(y)$ is compact set for every $y \in Y$.

Definition 2.2 Q is said to be residual set in Y,if Q contains the intersection of a sequence dense open set in Y;Q is dense in Y,if $Y$is a Baire space.

We also need the following well known fort theorem.

Lemma 2.1 [5]Let X be a metric space ,Y is a Baire space ,F is a usco mapping in Y, then there exists a dense residual set Q,for every $y \in Q$,F is lower semi-continuity in $y \in Y$.

Definition 2.3 Let Y is a problem space, $F(u)$ is the solution set to some problem $u \in Y$ and $x \in F(u)$

1) Let x is a essential set of u ,if for any open neighborhood O of x in X,there exists a open neighborhood U of u in Y,such that $F(u^*) \cap O \neq \phi$; for every$u^* \in U$;2) u is called weak-ed essential,if there exists a set of u is a essential set;3)u is called essential ,if any set of u is a essential set.

Let $u \in Y, x \in F(u)$, the summation set of every connected subset of x in $F(u)$ is said to be a connected component of $F(u)$(seeing[6]page 356),a connected component of $F(u)$ is a closed connected subset of $F(u)$,thereby a compact connected subset, if $u \neq u^*$, the connected components of $F(u)$ and $F(u^*)$ are either superposition or no-intersection, so $F(u)$ is de-compound a family of each other non-intersection summation set (finity or infinity),that is $F(u) = \cup_{\alpha \in \Lambda} F_\alpha$,thereinto Λ is a index set, $\alpha \in \Lambda$, F_α is a nonempty connected close subset,for every $\alpha, \beta \in \Lambda, \alpha \neq \beta$,and for every $F_\alpha \cap F_\beta = \phi$.

Definition 2.4 Let f is a nonempty connected close subset of solution set $F(u)$, f is said to be a essence set of $F(u)$ for any $u \in Y$,if for any open set O containing in f,there exists δ >0,such that $F(u^*) \cap O \neq \phi$ for every u^* that satisfy $\rho(u, u^*) < \delta$; If F_α is a essence set of $F(u)$,then F_α is said to be a essence connected section of $F(u)$;a essence set f of $F(u)$ is said to be a minimal essence set , if f is a minimal element in the family of essence sets of $F(u)$ according to inclusion of sets.

Definition 2.5[7]Let E a linear norm-ed space,X is any nonempty subset in E, E^* is conjugate space of E, $\langle \bullet, \bullet \rangle$ show the partnership of between E and E^*, let $S : X \to 2^X$ and $T : X \to 2^{E^*}$is set-valued mapping, then the problem of generalized quasi-variational inequalities indicate: find a point $\overline{y} \in S(\overline{y})$ and $u \in T(\overline{y})$such that : $Re\langle u, \overline{y} - x\rangle \leq 0, \forall x \in S(\overline{y})$.

Lemma 2.2 [8] Let E a linear norm-ed space , X is any non-empty close convex set in E , let $Q : X \to 2^X$ is a upper-continuity mapping with non-empty close value;
$J : X \times X \to R$ is a lowercontinuity mapping;$J(x, y)$ is convex for y;$\forall x \in X, J(x, x) \geq 0$; if for every sequence $z_n \to z$ and $y \in Q(z)$,there exists $y_n \in Q(z_n)$ such that

$$\overline{lim}_n J(z_n, y_n) \leq J(z, y)$$

then there exists $\bar{x} \in X$,such that $\bar{x} \in Q(\bar{x})$ and $J(\bar{x}, y) \geq 0, \forall y \in Q(\bar{x})$.

Lemma 2.3[8] Let E a linear norm-ed space,X is any nonempty close convex set in $E, T : X \to 2^{E^*}$ is nonempty close convex set and continuity,then $Re\langle w, y - x\rangle(w \in T(x))$ is a continual function in $X \times Y$.

3 the stability of the solution set of GQVI

Due to the existence [8]of the set to quasi-variational inequalities and lemma 2.2,lemma 2.3,we get out easily the following existence theorem (don't testify):

Theorem 3.1 Let E a linear norm-ed space,X is any nonempty close convex set in E, let $S : X \to 2^X$ is a upper-continuity mapping with nonempty close value, $T : X \to 2^{E^*}$ is nonempty close convex set and continuity. Then there exists $\overline{x} \in X$,such that 1)$\overline{x} \in S(\overline{x})$; 2)when $u \in T(\overline{x})$,get out $Re\langle u, \overline{x} - y\rangle \leq 0, \forall y \in S(\overline{x})$.

Let E a linear norm-ed space,X is any nonempty close convex set in E. Definition $B = \{(S, T) : S : X \to 2^X$ is a upper-continuity mapping with nonempty close value , $T : X \to 2^{E^*}$is nonempty close convex set and continuity$\}$, For every $b = (S, T), b^1 = (S^1, T^1) \in B$,we define

$$\rho_B(b, b^1) = \sup_{x \in X} h_1(T(x), T^1(x)) + \sup_{x \in X} h_2(S(x), S^1(x))$$

thereinto h_1, h_2 are Hausdorff metric in each other E^*, X.by all appearances (B, ρ_B) is a complete metric space.

For any $b = (S, T) \in B$,there exists a corresponding generalized quasi-variational inequalities problem ,so still marking with b and $E(b)$ is it's solution set .by theorem 3.1,we know easily $E(b) \neq \phi$, then we define a set-valued mapping $E : B \to 2^X$.

Lemma 3.1 $E : B \to 2^X$ is a usco mapping.

Proof: by X close and corollary 9 [5], only testifying $Graph(E)$ is a closed subset in $B \times X$,and

$$Graph(E) = \{((S,T),x) \in B \times X : x \in E((S,T))\}$$

for any a sequence $\{((S_n,T_n),x_n)\}_1^\infty \in Graph(E)$, and $((S_n,T_n),x_n) \to ((S,T),x)$, due to $S_n : X \to 2^X$ is a upper-continuity mapping with nonempty close value, adding to T_n continuity,getting out easily : $((S,T),x) \in B \times X$.
in the following testifying :$x \in E(S,T)$ because of $(S_n,T_n) \to (S,T), x_n \to x$ and $x_n \in E(S_n,T_n)$,by theorem 3.1,so: 1)$x_n \in S_n(x_n)$; 2) when $z_n \in T_n(x_n)$(might as well set $z_n \to z$),such that $Re\langle z_n, x_n - y\rangle \le 0, \forall y \in S_n(x_n)$. X close , $x_n \to x$ getting out $i)\forall\varepsilon > 0, \exists N_1, \forall n > N_1, d(x_n,x) < \varepsilon, \cdots\cdots\cdots(1)$
on the other hand, $S_n \to S \Rightarrow S_n(x_n) \to S(x_n).(\exists N_2, \forall n > N_2)$,S is upper-continuity,for the above ε, ordering $\delta = \varepsilon, S(x_n) \subset U(S(x),\varepsilon)$ $N = max(N_1,N_2)$, basing on $x_n \in S_n(x_n) \to S(x_n) \Longrightarrow x_n \in U(S(x_n),\varepsilon), y \in U(S(x_n),\varepsilon) \Longrightarrow x_n \in U(S(x),2\varepsilon)$,by formula(1)and the random ε,then $x \in S(x)$; ii) by T_n continuity and close value, $z_n \in T_n(x_n)$,then $z_n \to z$; as well as

$$d(z,(T(x))) \le d(z,z_n) + d(z_n,T_n(x_n)) + h(T_n(x_n),T_n(x)) + h(T_n(x),T(x)) \to 0$$

then $z \in T(x)$; $\langle\bullet,\bullet\rangle$ is continuity,there $|\langle z - z_n, x - y\rangle + \langle z_n, x - x_n\rangle| < \varepsilon$

$$\langle z - z_n, x - y\rangle + \langle z_n, x - x_n\rangle < \varepsilon$$

$$\langle z - z_n, x - y\rangle + \langle z_n, x - x_n - y + y\rangle < \varepsilon$$

that is

$$Re\langle z - z_n, x - y\rangle + Re\langle z_n, x - x_n - y + y\rangle < \varepsilon$$

$$Re\langle z, x - y\rangle - Re\langle z_n, x - y\rangle + Re\langle z_n, y - x_n\rangle + Re\langle z_n, x - y\rangle < \varepsilon$$

$$Re\langle z, x - y\rangle < Re\langle z_n, x_n - y\rangle + \varepsilon \le \varepsilon,$$

(reasoning :$Re\langle z_n, x_n - y\rangle \le 0, \forall y_n \in S(x_n)$), due to the random ε, $S(x_n) \subset U(S(x),\varepsilon)$ and $y \in U(S(x_n),\varepsilon)$,there:

$$Re\langle z, x - y\rangle \le 0, \forall y \in S(x)$$

summing:

$$x \in (S,T).$$

which completes the proof.

Since definition 2.1 and 2.3,knowing easily:

Lemma 3.2 b is essence, the sufficient and necessary condition of which is that A set-valued mapping $E : B \to 2^X$ is lower-continuity in b .

The important conclusion:

Theorem 3.2 there exists a dense residual set Q in B,for every $b \in Q, b$ is essence.

Proof: by lemma 2.1 and 3.1,3.2,the theorem is proved easily .

So, we obtain the generalized quasi-variational inequalities is stability in the dense residual set, that is ,basing on Baire classify, the most absolutely of the generalized quasi-variational inequalities are stability.

4 on essential components of the solution set of GQVI

Theorem 4.1 for each $b \in B$,there exists a essential components of $E(b)$.

In order to prove the theorem, we firstly present the following condition $((A)$: let $B, X$are metric spaces (the defined before),$E : B \to 2^X$is a mapping, for any two nonempty closed sets K_1, K_2 in X, $K_1 \cap K_2 = \phi$ and for any two points $b_1, b_2$of B, if $E(b_1) \cap K_1 = \phi, E(b_2) \cap K_2 = \phi$. then there exists $b^* \in B$, such that
$d(b^*, b_2) \leq d(b_1, b_2), d(b_1, b^*) \leq d(b_1, b_2)$, and $E(b^*) \cap (K_1 \cup K_2) = \phi$ by theorem 3.1[3],if E satisfy (A),then
(1)for any $b \in B$,then there exists a minimal essential set of $E(b)$, and each minimal essential set of $E(b)$is connected.
(2) for each $b \in B$, there exists at least a essential component of $E(b)$.

In order to prove theorem 4.1.so we testify condition (A) :

Proof: for any nonempty closed K_1, K_2in X,$K_1 \cap K_2 = \phi$ and any point b_1, b_2in B,if $E(b_1) \cap K_1 = \phi, E(b_2) \cap K_2 = \phi$,firstly we construct the new $b^* = (S^*(x), T^*(x))$,seeing

$$S^*(x) = \lambda(x)S_1(x) + \mu(x)S_2(x)$$

$$T^*(x) = \lambda(x)T_1(x) + \mu(x)T_2(x)$$

thereinto $b_1 = (S_1, T_1), b_2 = (S_2, T_2)$

$$\lambda(x) = \frac{d(x, K_2)}{d(x, K_1) + d(x, K_2)}$$

$$\mu(x) = \frac{d(x, K_1)}{d(x, K_1) + d(x, K_2)}$$

by all appearances $\lambda(x), \mu(x)$continuum, $\lambda(x) \geq 0, \mu(x) \geq 0$,and $\lambda(x) + \mu(x) = 1$.testify easily $b^* = (S^*, T^*) \in B$ then $\forall x \in X$,

$$h_1(S^*, S_1) \leq h_1(S_1, S_2), \quad h_1(S^*, S_2) \leq h_1(S_1, S_2)$$

$$h_2(T^*, T_1) \leq h_2(T_1, T_2), \quad h_2(T^*, T_2) \leq h_2(T_1, T_2)$$

so :

$$\sup h_1(S^*, S_1) \leq \sup h_1(S_1, S_2), \quad \sup h_1(S^*, S_2) \leq \sup h_1(S_1, S_2)$$

$$\sup h_2(T^*, T_1) \leq \sup h_2(T_1, T_2), \quad \sup h_2(T^*, T_2) \leq \sup h_2(T_1, T_2)$$

$$\sup h_1(S^*, S_1) + \sup h_2(T^*, T_1) \leq \sup h_1(S_1, S_2) + \sup h_2(T_1, T_2)$$

$$\sup h_1(S^*, S_2) + \sup h_2(T^*, T_2) \leq \sup h_1(S_1, S_2) + \sup h_2(T_1, T_2)$$

$\Longrightarrow$

$$\rho_B(b^*, b_1) \leq \rho_B(b_1, b_2)$$

$$\rho_B(b^*, b_2) \leq \rho_B(b_1, b_2)$$

the following we testify $E(b^*) \cap (K_1 \cup K_2) = \phi$
$i)$if $x \in K_1$,then $\lambda(x) = 1, \mu(x) = 0$,so

$$S^* = \lambda(x)S_1(x) + \mu(x)S_2(x) = S_1$$

$$T^* = \lambda(x)T_1(x) + \mu(x)T_2(x) = T_1$$

then $b_1 = (S_1, T_1) = b^* = (S^*, T^*)$, by $E(b_1) \cap K_1 = \phi \Longrightarrow x \notin E(b_1) = E(b^*)$
$ii)$if $x \in K_2$,then $\lambda(x) = 0, \mu(x) = 1$,so

$$S^* = \lambda(x)S_1(x) + \mu(x)S_2(x) = S_2$$

$$T^* = \lambda(x)T_1(x) + \mu(x)T_2(x) = T_2$$

then $b_2 = (S_2, T_2) = b^* = (S^*, T^*)$. by $E(b_2) \cap K_2 = \phi \Longrightarrow x \notin E(b_2) = E(b^*)$; that is $E(b^*) \cap (K_1 \cup K_2) = \phi$.which the condition $(A)$is satisfied. that is for each $b \in B$,there exists a essential components of $E(b)$. which completes the proof.

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