Applied Mathematical Sciences, Vol. 2, 2008, no. 10, 463 - 469

Similarity Reductions of Benjamin-Bona-Mahony Equation

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Abstract

In this paper, we firstly study the classical Lie symmetries of the Benjamin-Bona-Mahony (BBM) equation which is obtained through the Lie group method of infinitesimal transformations. Secondly using the classical symmetries of the equation, similarity reductions are obtained and it is shown that one of these similarity reductions has no the Painlevé property.

Mathematics Subject Classification: 35A30, 11F22, 17B10

Keywords: Integrability of nonlinear partial differential equations, BBM equation, Lie symmetries

1 Introduction

The Benjamin-Bona-Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1}$$

like the Korteweg-de Vries (KdV) equation

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{2}$$

was originally derived as approximation for surface water waves in a uniform channel [1,2].

Both (1) and (2) also cover cases of the following type: surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, etc. The wide applicability of these equations is the main reason why, during the last decades, they have attracted so much attention from mathematicians.

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The main mathematical difference between KdV and BBM models can be most readily appreciated by comparing the dispersion relation for the respective linearized equations. It can be easily seen that these relations are comparable only for small wave numbers (i.e., long waves) and they generate drastically different responses to short waves (which are irrelevant to its role as a physical model). This is one of the reasons why, whereas existence and regularity theory for the KdV equation is difficult, the theory of the BBM equation is comparatively simple. The computing is also much easier for (1) than for (2).

The application of Lie transformations group theory for the construction of solutions of nonlinear partial differential equations (PDEs) is one of the most active fields of research in the theory of nonlinear PDEs and applications. The fundamental basis of the technique is that when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. In order to determine solutions of PDE (1) that are not equivalent by the action of group, we must calculate the one dimensional optimal system. Most of the required theory and description of method can be found in [3,4,5,6].

In Section 2, we find the Lie symmetry algebra of the BBM equation and present the optimal systems of one dimensional subalgebras of Lie symmetry algebra. In Section 3, we use these subalgebras to perform similarity reductions and to obtain the similarity solutions. In Section 4, we give the Painlevé analysis for ordinary differential equations (ODEs) and apply to the similarity reduction of the BBM equation. Some conclusions are drawn in Section 5.

2 Lie Symmetries and Optimal Systems

2.1 Lie Symmetries

In this sub section, we want to present the most general Lie group of point transformations, which leaves BBM equation (1) invariant.

To apply the classical Lie symmetry group method to BBM equation (1), we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u). The associated Lie algebra of the infinitesimal system involves the set of vector fields of the form

$$\mathbf{v} = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u}. \tag{3}$$

The symmetry condition

$$\mathbf{pr}^{(3)}\mathbf{v}\triangle|_{\triangle} = 0. \tag{4}$$

yields an overdetermined system of PDE for the unknown functions $\xi^x(x, t, u)$, $\xi^t(x, t, u)$ and $\eta^u(x, t, u)$ where $\triangle$ is the manifold defined by (1) in jet space

$J^{(3)}_{x,t;u}$ and $\mathbf{pr}^{(3)}\mathbf{v}$ is the third prolongation of $\mathbf{v}$. By solving this system, we found the unknown functions ξ^x, ξ^t and η^u. The Lie algebra admitted by (1) is

$$L_3 = \left\{ v_1 = \frac{\partial}{\partial x},\, v_2 = \frac{\partial}{\partial t},\, v_3 = t\frac{\partial}{\partial t} - (1+u)\frac{\partial}{\partial u} \right\}. \tag{5}$$

2.2 Optimal System of Lie Symmetries

We define a relation between two invariant solutions to hold true if the first one can be mapped to the other by applying a transformation group generated by a linear combinations of the operators in (5). Since these mappings are reflexive, symmetric and transitive, the relation is an equivalence relation, which induces a natural partition on the set of all group invariant solutions into equivalence classes. We need only present one solution from each equivalence class, as the rest may be found by applying appropriate group symmetries; a complete set of such solutions is referred to as an "optimal system" of group invariant solutions.

The problem of deriving an optimal system of group invariant solutions is equivalent to find an optimal system of Lie symmetries (or subalgebras spanned by these operators). The method used here is given by Olver in [4], which basically consists of taking linear combinations of the generators in (5), and reducing them to their simplest equivalent form by applying carefully chosen adjoint transformations.

$$\mathrm{Ad}(\exp(\epsilon\, v_i))v_j = v_j - \epsilon\,[v_i, v_j] + \tfrac{1}{2}\epsilon^2[v_i,[v_i,v_j]] - \cdots .$$

Here $[v_i, v_j]$ is the usual commutator, given by

$$[v_i, v_j] = v_i v_j - v_j v_i \ .$$

For brevity we omit the details, and just state the result that an optimal system of generators is

$$\{L_{1,1} = \{v_1\}\,,\, L_{1,2} = \{\alpha v_1 + v_2\}\,,\, L_{1,3} = \{\alpha v_1 + v_3\}\}\,, \tag{6}$$

where α denote arbitrary real constant.

3 Reductions to ODEs and exact solutions

In this section we use the method of characteristics to determine the invariants and reduced ODEs corresponding to each subalgebra given in (6).

Symmetry variables and the invariants of the subalgebras of the Lie algebra L_3 are given in table 1. The result of this can be summarized as follows, where ξ is the symmetry variable, $F(\xi)$ is invariant function related to u, and have to be determined using the reduced ODEs

Subalgebra	Symmetry variable	Function $u(x,t)$
$L_{1,1}$	$\xi = t$	$u = F(\xi)$
$L_{1,2}(\alpha)$	$\xi = \alpha t - x$	$u = F(\xi)$
$L_{1,3}(\alpha)$	$\xi = e^x t^{-\alpha}$	$u = \frac{F(\xi)}{t} - 1$

Table 1
Invariants of the subalgebras of the Lie algebra L_3

3.1 Solutions

3.1.1 $L_{1,1} = \{v_1\} = \left\{\frac{\partial}{\partial x}\right\}$

The reduced equation of $L_{1,1}$ is

$$F' = 0 \tag{7}$$

and gives the constant solutions of (1).

3.1.2 $L_{1,2}(\alpha) = \{\alpha v_1 + v_2\} = \left\{\alpha\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right\}$

The reduced equation for the subalgebra $L_{1,2}(\alpha)$ is

$$\alpha F''' + (\alpha - 1)F' - FF' = 0. \tag{8}$$

For $\alpha \neq 0$, equation (8) integrating twice with respect to ξ, we obtain

$$\alpha(F')^2 = \frac{1}{3}F^3 - (\alpha - 1)F^2 + c_1 F + c_2. \tag{9}$$

where c_1 and c_2 denote arbitrary constants. Choosing $c_2 = 0$, we find the solution of this equation as follows.

$$F(\xi) = \beta^2 sn^2\left(\sqrt{\frac{3\alpha}{\gamma}}\,\xi\right), \tag{10}$$

where

$$\beta = \tfrac{1}{2}\left(\sqrt{9(\alpha-1)^2 - 4c_1} - 3(\alpha-1)\right),$$

$$\gamma = -\tfrac{1}{2}\left(\sqrt{9(\alpha-1)^2 - 4c_1} + 3(\alpha-1)\right)$$

and sn denotes Jacobi's elliptic function [7].

3.1.3 $L_{1,3}(\alpha) = \{\alpha v_1 + v_3\} = \left\{\alpha\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - (1+u)\frac{\partial}{\partial u}\right\}$

The reduced equation for the subalgebra $L_{1,3}(\alpha)$ is

$$\alpha\xi^3 F''' + (3\alpha+1)\xi^2 F'' + (2\alpha+1)\xi F' - \xi F F' + F = 0. \qquad (11)$$

Changing the variable as $\tau = \ln\xi$, we obtained

$$\alpha F''' + F'' + \alpha F' + F - FF' = 0. \qquad (12)$$

4 Painlevé Analysis for ODEs

According to the Ablowitz, Ramani and Segur (ARS) conjecture [8], *every ordinary differential equation obtained by an exact reduction of a nonlinear partial diferential equation solvable by Inverse Scattering Transform (IST) method, has the Painlevé property.* This conjecture therefore provides a necessary condition for the integrability of a given partial differential equation.

The Painlevé property for ODEs is defined as follows. The solutions of a system of ODEs are regarded as (analytic) functions of a complex variable. The "movable" singularities of the solution are the singularities of the solution (as a function of τ) whose location depends on the initial conditions and are, hence, movable. (Fixed singularities occur at points where the coefficients of the equation are singular). The ODE system is said to possess the Painlevé property when all the movable singularities are single-valued (simple poles) [9].

The ARS algorithm was developed in order to determine whether or not a nonlinear ODE (or a system of ODE's) admits movable branch points, either algebraic or logarithmic. It is important to keep in mind that the occurrence of movable essential singularities can not be detected by this procedure.

The ARS algorithm proceeds in three steps, dealing with the dominant behaviors, the resonances, respectively [8,10].

To applied The Painlevé analysis following to [8,10] for the equation (12), we look for a solution of (12) in the form

$$F = F_0(\tau-\tau_0)^\beta + O\left((\tau-\tau_0)^{\beta-1}\right) \qquad (13)$$

where τ_0 is arbitrary. Substituting (13) in to (12) shows that for certain values of β, two or more terms in the equation may balance (depending on F_0), and the rest can be ignored as $\tau \to \tau_0$. For each such choice of β, the terms which can balance are called the leading terms. Requiring that the leading terms do balance (usually) determines F_0.

Here for $\alpha = 0$ we find $\beta = -1$ and $F_0 = -2$, but for $\alpha \neq 0$ we find $beta = -2$ and $F_0 = 12\alpha$. The corresponding resonances can be found as for $\alpha = 0$, $n_1 = -1, 2$ and for $\alpha \neq 0$, $n_2 = -1, 4, 6$. At the resonances $n_1 = 2$ and $n_2 = 4$, the compatibility conditions can not be satisfied and therefore the equation (12) does not pass the Painlevé test.

5 Conclusions

Using the Lie group method, we obtained the similarity reductions and solutions to the BBM equation. Considering the one of the similarity reduction, we showed that it does not pass the Painlevé test. Acording to the ARS conjecture BBM equation is not solvable by IST.

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Received: August 16, 2007