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Application of He's Variational Iteration Method for Solving Seventh Order Sawada-Kotera Equations

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Abstract

In this paper, He's variational iteration method (VIM) has been used to obtain solutions of the seventh-order Sawada-Kotera equation (sSK) and a Lax's seventh order KdV equations(LsKdV).The numerical solutions are compared with the Adomian decomposition method(ADM) and the known analytical solutions.The work confirms the power of the VIM in reducing the size of calculations w.r.t. ADM. Some illustrative examples have been presented.

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1 Introduction

Analytical methods commonly used to solve nonlinear equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors.

Recently introduced variational iteration method by He[5, 6, 7, 8], which gives rapidly convergent successive approximations of the exact solution if such a solution exists, has proven successful in deriving analytical solutions

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of linear and nonlinear differential equations. This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory. He [6, 7, 13] has applied this method for obtaining analytical solutions of autonomous ordinary differential equation, nonlinear partial differential equations with variable coefficients and integro-differential equations. The variational iteration method was successfully applied to Burger's and coupled Burger's equations [1], to Schruodinger-KdV, generalized KdV and shallow water equations [2], to linear Helmholtz partial differential equation [11]. Linear and nonlinear wave equations, KdV, K(2,2), Burgers, and cubic Boussinesq equations have been solved by Wazwaz [14, 15] using the variational iteration method.

In the present paper we employ VIM method for solving following equations.

$$u_t + (63u^4 + 63(2u^2u_{xx} + uu_x^2) + 21(uu_{xxxx} + u_{xx}^2 + u_xu_{xxx}) + u_{xxxxxx})_x = 0, \quad (1)$$

$$u_t + (35u^4 + 70(u^2u_{xx} + uu_x^2) + 7(2uu_{xxxx} + 3u_{xx}^2 + 4u_xu_{xxx}) + u_{xxxxxx})_x = 0, \quad (2)$$

Eq. (1) is known as the seventh order Sawada -Kotera equation [4, 9] and Eq. (2) is known as the Lax's seventh-order KdV equation [4, 12] respectively. Further we compare the result with given solutions using ADM [4, 3]. The paper has been organized as follows. Section II, gives a brief review of VIM. Section III, consists of main results of the paper, in which variational iteration method of the sSK and LsKdV equations has been developed. In Section IV, illustrative examples are given. Conclusions are presented in Section V.

2 He's variational iteration method

For the purpose of illustration of the methodology to the proposed method, using variational iteration method, we begin by considering a differential equation in the formal form,

$$Lu + Nu = g(x,t), \quad (3)$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional as follow

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}(\xi) - g(\xi)\right) d\xi, \quad n \geq 0, \quad (4)$$

where λ is a general Lagrangian multiplier [10], which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, $\tilde{u}_n$ is considered as a restricted variation [7, 8, 10] i.e., $\delta\tilde{u}_n = 0$.

So, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Consequently, the solution

$$u(x,t) = \lim_{n \longrightarrow \infty} u_n(x,t). \tag{5}$$

3 Applying VIM for sSK and LsKdV

Now for applying VIM, first we rewrite Eq. (1) in the following form

$$L_t(u) + (63N_1(u) + 63(2N_2(u) + N_3(u)) + 21(N_4(u)) + L_x(u))_x = 0, \tag{6}$$

where the notations $N_1(u) = u^4$, $N_2(u) = u^2 u_{xx}$, $N_3(u) = uu_x^2$,
$N_4(u) = uu_{xxxx} + u_{xx}^2 + u_x u_{xxx}$, symbolize the nonlinear terms, respectively. The notation $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial^6}{\partial x^6}$ symbolize the linear differential operators. The correction functional for Eq.(6) reads

$$u_{n+1}(x,t) \;=\; u_n(x,t) + \int_0^t \lambda(\xi) \left[\frac{\partial}{\partial \xi}(u_n) + (N(\tilde{u}_n))_x \right] d\xi, \quad n \geq 0, \tag{7}$$

where, $N(u) = (63N_1(u) + 63(2N_2(u) + N_3(u)) + 21(N_4(u)) + L_x(u)$. Taking variation with respect to the independent variable u_n, noticing that $\delta N(\tilde{u}_n) = 0$

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \left[\frac{\partial}{\partial \xi}(u_n) + (N(\tilde{u}_n))_x \right] d\xi \\ &= \delta u_n(x,t) + \lambda\, \delta u_n|_{\xi=t} - \int_0^t \lambda'(\xi)\, \delta u_n d\xi = 0, \end{aligned} \tag{8}$$

This yields the stationary conditions

$$1 + \lambda(\xi) = 0, \qquad \lambda'(\xi)|_{\xi=t} = 0. \tag{9}$$

This in turn gives $\lambda(\xi) = -1$. Substituting this value of the Lagrange multiplier into the functional (7) gives the iteration formula

$$u_{n+1}(x,t) \;=\; u_n(x,t) - \int_0^t \left[\frac{\partial}{\partial \xi}(u_n) + (N(u_n))_x \right] d\xi, \quad n \geq 0. \tag{10}$$

Using the zeroth approximation $u_0(x,t)$ into (10) we obtain the successive approximations.

In the same manner for LsKdV (2) we got the following iteration formula

$$u_{n+1}(x,t) \;=\; u_n(x,t) - \int_0^t \left[\frac{\partial}{\partial \xi}(u_n) + (F(u_n))_x \right] d\xi, \quad n \geq 0, \tag{11}$$

where $F(u) = 35u^4 + 70(u^2 u_{xx} + uu_x^2) + 7(2uu_{xxxx} + 3u_{xx}^2 + 4u_x u_{xxx}) + u_{xxxxxx}$.

Finally, we approximate the solution $u(x,t) = \lim_{n \longrightarrow \infty} u_n(x,t)$.

4 Illustrative Examples

To demonstrate the effectiveness of the method we consider here Eqs.(1) and (2) with given initial condition.

Example 4.1 *)[4] Consider the sSK equation (1) with the initial condition*

$$u(x,0) = \frac{4k^2}{3}(2 - 3\tanh^2(k\,x)). \tag{12}$$

Substituting (12) into Eq.(10) we obtain the following successive approximations

$$\begin{aligned}
u_0(x,t) &= \frac{4}{3}k^2\left(2 - 3\tanh^2(kx)\right),\\
u_1(x,t) &= u_0(x,t) + \frac{1}{9}k^8 \operatorname{sech}^2(kx)\, t[2176 - 896\cosh(2kx)],\\
u_2(x,t) &= u_1(x,t) + \frac{1}{27}\operatorname{sech}^8(kx)k^{14}\, t^2([6328576 - 6566144\cosh(2kx) +\\
&\quad 1077248\cosh(4kx) + 24832\cosh(6kx) - 12544\cosh(8kx)] +\\
&\quad \frac{1}{3}k^6\, t[2544812032 - 2746548224\cosh(2kx) + 305070080\cosh(4kx) +\\
&\quad 41746432\cosh(6kx) - 5619712\cosh(8kx)] + \frac{1}{9}k^{12}\, t^2[50980192256 -\\
&\quad 23855104000\cosh(2kx) - 55593402368\cosh(4kx) +\\
&\quad 17983078400\cosh(6kx) - 1258815488\cosh(8kx)]\\
&\quad \frac{1}{135}k^{18}t^3[-238459436400640 + 291359575506944\cosh(2kx)\\
&\quad -82083401695232\cosh(4kx) + 10956730007552\cosh(6kx) -\\
&\quad 563949338624\cosh(8kx)])
\end{aligned}$$

and so on. In Fig.1 and Fig. 2 we draw $u_3(x,t)$ and $u(x,t) = \frac{4k^2}{3}(2 - 3\tanh^2(k(x - \frac{256k^6}{3}t)))$ which is the exact solution [12] for $k = 0.1$ and $-100 < x < 100$.

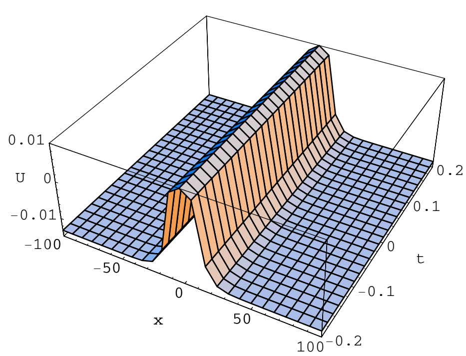

Fig. 1. Approximate Solution $u_3(x,t)$

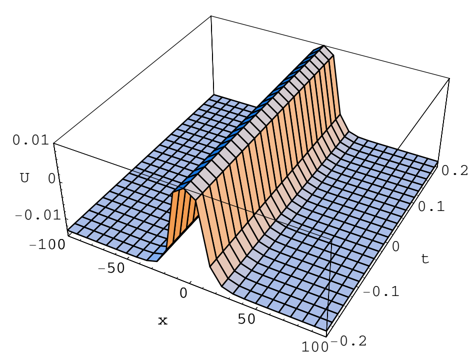

Fig. 2. Exact Solution $u(x,t)$

Remark 1: sSK equation has been solved by ADM by El-Sayed and Kaya [4]. It should be remarked that the graph drawn here using VIM is agreement with that drawn using ADM [4] but just after 3 iteration.
Remark 2: In [4], for solving this equations using ADM they compute Adomian polynomials for $N_1(u)$, $N_2(u)$, $N_3(u)$ and $N_4(u)$.

Example 4.2 *)[4] Consider the LsKdV equation with given initial condition,*

$$u(x,0) = 2k^2 sech^2(kx). \tag{13}$$

Substituting (13) into Eq.(11) we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= 2k^2\text{sech}^2(kx), \\ u_1(x,t) &= u_0(x,t) - 128k^8t\text{sech}^2(kx), \\ u_2(x,t) &= u_1(x,t) - \frac{128}{3}\text{sech}^8(kx)k^{14}t^2(44040192t^3k^{18} + 1720320t^2k^{12} + \\ &\quad 30464tk^6 + (-3440640t^2k^{12} - 7168tk^6 + 1803)\cosh(2kx) + \\ &\quad 36(448k^6t - 11)\cosh(4kx) - 3\cosh(6kx) - 2004) \end{aligned}$$

and so on. Using the above terms, in Fig.3, $u_3(x,t)$ is drawn for $k = 0.1$ and $x \in [-100, 100]$. In Fig.4, exact solution $u(x,t) = 2k^2\text{sech}^2(kx)$ [12] is drawn.

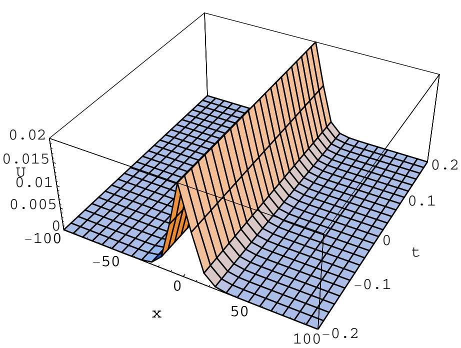

Fig. 3.Approximate Solution $u_3(x,t)$

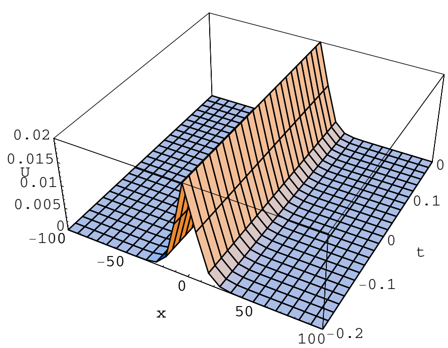

Fig. 4. Exact Solution$u(x,t)$

Remark 3: It should be remarked that the graph drawn here using VIM is in excellent agreement with that drawn using ADM [4]but just after 3 iteration.
Remark 4: In [4], for every nonlinear parts of $F(u)$ they have calculated Adomian polynomials.

5 Conclusion

Variational iteration method is a powerful tool which is capable of handling linear/nonlinear partial differential equations. The method has been successfully

applied to sSK and LsKdV equations. Also, comparisons were made between He's variational iteration method and Adomian decomposition method (ADM) for sSK and LsKdV equations. The VIM reduces the volume of calculations without requiring to compute the Adomian polynomials. However, ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work. For nonlinear equations that arise frequently to express nonlinear phenomenon, He's variational iteration method facilitates the computational work and gives the solution rapidly if compared with Adomian method.
Mathematica has been used for computations in this paper.

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