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# Bilaleral Contact with Tresca's Friction Law with Internal State Variables

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**Abstract**

The subject of this work is the study of a value problem describing the quasistatic evolution of semilinear retetype viscoplastic models with internal state variables, and we suppose the problem of Tresca's Friction Law at the presence of recal forces.

The existence and uniqueness of the solution is proved using results of evolutionary variational inequalities and a fixed point theorem.

**Mathematics Subject Classification:** 74M15, 74S05, 65M60

**Keywords:** viscoplasticity, Trasca's law, variational inequality, fixed point

## 1 Introduction

For the bilateral contact problem studied in this paper, the friction is modelled by tesca's law with intarnal state variables.
So, in section 2 contains the basic notations on the functional spaces and some hypothesis used in the following .
Section 3, mechanical problem a fixed is stated together with his variational formulation, in section 4 using results of evolutionary variational inequalities and a fixed point theorem , an existence and uniqueness result is obtained.

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## 2 Statement of the problem.Hypothesis

We assume the contact is bilateral, i.e. there is no loss of the contact during the process. Thus, the normal displacement $u_\nu$ vanishes on $\Gamma_3$ at any time. We model the friction with the tresca friction law.
Let $\Omega \subset \mathbb{R}^N (N = 1, 2, 3)$ a bounded domain and whose boundary $\Gamma$ assumed to be sufficiently smooth, is partitionned into three disjoint measurable parts $\Gamma_1$ ,$\Gamma_2$ and $\Gamma_3$ . Let meas $\Gamma_1 > 0$, let $T > 0$ be a time .
Let M be a natural number; we cosider the follawing mixed problem:

**(1)** $\dot{\sigma} = \xi(\varepsilon(\dot{u})) + G(\sigma, u, k)$ in $\Omega \times [0, T]$

**(2)** $\dot{k} = \phi(\varepsilon(u), \sigma, k)$ in $\Omega \times [0, T]$

**(3)** $Div\sigma + f_0 = 0$ in $\Omega \times [0, T]$

**(4)** $u = 0$ on $\Gamma_1 \times [0, T]$

**(5)** $\sigma\nu + \Phi(x)u = f_2$ on $\Gamma_2 \times [0, T]$

**(6)**
$$\left.\begin{array}{c} u_\nu = 0, \ \|\sigma_\tau\| \leq g, \\ \|\sigma_\tau\| < g \Rightarrow \dot{u}_\tau = 0, \\ \|\sigma_\tau\| = g \Rightarrow \exists\lambda \ > \sigma_\tau = -\lambda\dot{u}_\tau \end{array}\right\} \text{ on } \Gamma_3 \times [0, T]$$

**(7)** $u(0) = u_0, \ \sigma(0) = \ \sigma_0, \ k(0) = k_0$ in $\Omega$

in which the unknowns are the functions
$u : \Omega \times (0 \ , T \ ) \to \mathbb{R}^N$, $\sigma : \Omega \times (0, T) \to S_N$ and $k : \Omega \times (0, T) \to \mathbb{R}^M$
in which $k$ may be interpreted as an internal state variable and $\xi$ , $k$ and $G$ are given constituve functions.
In (1)-(7) $u$ represents the displacement function, $\sigma$ represents the sress function, $\varepsilon(u)$ denotes the small strain tensor,$f_0$ is given body force, $f_2$ is given boundary data, and $u_0$, $\sigma_0$ and $k_0$ are the initial data. In (6), $g \geqslant 0$ is the friction bound function, i.e magnitude of the limiting friction traction at which slip begins. The strict inequality in (6) holds in the stick zone and the equality holds in the slip zone. The equation (5) means that the Cauchy vector $\sigma\nu$ is proportional on the displacement.
Viscoplastic models of the form (1),(2) are used in order to model the behaviour of real bodies for which the plastic rate of deformation depends also on an internal state variable.

Some of the internal state variables considered by many authors are the plastic strain, a number of tensor variables that takes into account the spatial display of dislocation for the internal state variables. Here we suppose that $k$ is a vector-valued function which satisfiers (2) were $\phi$ is agiven function.
In order to obtain variatinal formulations for the problem (1)-(7), let us consider the following notations:

$$H = \left\{ v = (v_i) / v_i \in \mathbb{L}^2(\Omega),\ i = \overline{1, N} \right\}$$
$$H_1 = \left\{ v = (v_i) / v_i \in H^1(\Omega),\ i = \overline{1, N} \right\}$$
$$V = \{ v \in H_1 : v = 0 \text{ a.e on } \Gamma_1 \}$$
$$V_{\ 1} = \{ v \in V / \ v_\nu = 0 \text{ a.e on } \Gamma_3 \}$$
$$Q = \{ \tau = (\tau_{ij}) : (\tau_{ij}) = (\tau_{ji}) \in \mathbb{L}^2(\Omega),\ 1 \leq i, j \leq p \}$$
$$Q_1 = \{ \tau \in Q, \quad Div\ \tau \in H \}$$

The spaces $H$, $H_1$, $V$, $V_1$, $Q$, $Q_1$ are real Hilbert spaces endowed with the canonical inner products denoted by $\langle ., \ . \rangle_H$, $\langle ., \ . \rangle_{H_1}$, $\langle ., \ . \rangle_V$ , $\langle ., \ . \rangle_{V_1}$, $\langle ., \ . \rangle_Q$, $\langle ., \ . \rangle_{Q_1}$ respectively.
We assume that the force and traction satisfy

**(8)** $f_0 \in W^{1,\infty}(0, T, H)$, $f_2 \in W^{1,\infty}(0, T, H_{\Gamma_2})$.

and the friction bound satisfies

**(9)** $g \in L^\infty(\Gamma_3)$, $g \geqslant 0$ on $\Gamma_3$

Denote by $f(t)$ the element of $V_1$ giving by

**(10)** $\langle f(t), v \rangle_V = \int_\Omega f_0 . v \ ds + \int_{\Gamma_2} f_2 . v da \ \forall v \in V_1$

For all $t \in [0, T]$ let $j : V_1 \to \mathbb{R}_+$ be the functional

$$j(v) = \int_{\Gamma_2} g \, \|v_\tau\| \, da.$$

For all $t \in [0, T]$, we note that conditions (8) and (9) imply

**(11)** $$f \in W^{1,\infty}(0, T, V_1)$$

and

**(12)** $$j(v) \leq c \, \|g\|_{\mathbb{L}^\infty(\Gamma_3)} \, \|v\| \qquad \forall v \in V_1$$

We also introduce the notation $\Sigma(t)$ defined by:

**(13)** $\Sigma(t) = \Big\{\tau \in Q: \quad \langle \tau, \varepsilon(v)\rangle_Q + \langle \Phi u_0, v\rangle_{L^2(\Gamma_2)} + j(v) \geq \ \langle f(t), v\rangle_V, \forall v \in V_1\Big\},$
$t \in \overline{[0,T]}$

Finally, assume that the initial data satisfy

**(14)** $$u_0 \in V_1,\ \sigma_0 \in \Sigma(0)$$

We have the following result.

# 3 Variational formulation

**Lemma 1** *If $u$ and $\sigma$ are regular functions satisfying (3)-(6) then for all $t \in [0,T]$,*

**(15)** $u(t) \in V_1, \quad \langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_Q + \langle \Phi u(t), v - \dot{u}(t)\rangle_{\mathbb{L}^2(\Gamma_2)^N} +$
$$j(v) - j(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_V \quad \forall v \in V_1$$

**Proof:** Let $t \in [0,T]$ and $v \in V_1$ be an arbitrary test function. Multiplying the equation (3) by $v - \dot{u}(t)$ and integrating by parts, we obtain:

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_Q = \int_\Omega f_0(t).(v - \dot{u}(t))ds + \int_\Gamma \sigma\nu.(v - \dot{u}(t))da$$

Using the boundary condition (4) and (5) we can replace the right band side of the above equality by

$$\langle f(t), v - \dot{u}(t)\rangle_V - \int_{\Gamma_2} \Phi u.(v - \dot{u}(t))da + \int_{\Gamma_3} \sigma\nu.(v - \dot{u}(t))da$$

From the contact boundary condition (6) we derive the relation

$$\sigma_\tau(t).\dot{u}_\tau(t) = -g\,\|\dot{u}_\tau(t)\| \qquad on \Gamma_3$$

Thus, on $\Gamma_3$,

$$\begin{aligned}\sigma\nu.(v - \dot{u}(t)) &= \sigma_\nu(t).(v_\nu - \dot{u}_\nu(t)) + \sigma_\tau(t).(v_\tau - \dot{u}_\tau(t))\\ &= \sigma_\tau(t).v_\tau - \sigma_\tau(t).\dot{u}_\tau(t)\\ &= \sigma_\tau(t).v_\tau + g\,\|\dot{u}_\tau(t)\|\\ &\geq g(\|\dot{u}_\tau(t)\| - \|v_\tau\|).\end{aligned}$$

Therefore,

$$\int_{\Gamma_3} \sigma\nu.(v - \dot{u}(t))da \geq j(\dot{u}(t)) - j(v),$$

and we have

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + \langle \Phi w, v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} \geq \langle f(t), v - \dot{u}(t) \rangle_V + j(\dot{u}(t) - j(v).$$

i.e the inequality in (12) holds.
The previous lemma leads us to consider the following weark formulations of the frictional problem $(1) - (7)$ .

**Problem 2** Find a displacement field $u : [0,T] \to \mathbb{V}_1$, a stress field $\sigma : [0,T] \to Q_1$and $k : [0,T] \to \mathbb{R}^M$

**(16)** $$\dot{\sigma}(t) = \xi(\varepsilon(\dot{u})) + G(\sigma, u, k) \quad \text{a.e } t \in (0,T)$$

**(17)** $$\dot{k} = \phi(\varepsilon(u), \sigma, k)$$

**(18)** $$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + \langle \Phi u(t), v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} + j(v) - j(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle_V \quad \forall v \in V_1$$

$$u(0) = u_0, \ \sigma(0) = \ \sigma_0, k(0) = k_0$$

# 4 Existence and Uniqueness results

In the study of the problem (1)-(7), we consider the following assuptions:

**(19)**
$$\begin{cases} \xi : \Omega \times S_N \to S_N \\ (\mathbf{a}) \quad \xi_{ijkh} \in L^\infty(\Omega) \text{ for all } \ i,j,k,h = \overline{0,N} \\ (\mathbf{b}) \quad \xi\sigma.\tau = \xi.\sigma\tau \ \forall \sigma, \tau \in S_N. \\ (\mathbf{c}) \quad \exists \alpha > 0 / \xi\sigma.\sigma \geqslant \alpha \left|\sigma\right|^2 \ \forall \sigma \in S_N. \end{cases}$$

**(20)**
$$\begin{cases} G : \Omega \times S_N \times S_N \times \mathbb{R}^M \to S_N \\ (\mathbf{a}) \quad \exists \tilde{L} > 0 \text{ that } \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, \ k_1, k_2 \in \mathbb{R}^M \text{ a.e in } \Omega \\ |G(x, \sigma_1, \varepsilon_1, k_1) - G(x, \sigma_2, \varepsilon_2, k_2)| \leq \tilde{L}(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |k_1 - k_2|) \\ (\mathbf{b}) \quad x \to G(x, \sigma, \varepsilon, k) \text{ is mesurable function with respect to the Lebesgue measure on } \Omega \text{ for all } \sigma, \varepsilon \in S_N, \ k \in \mathbb{R}^M \\ (\mathbf{c}) \quad x \to G(x, 0, 0, 0) \in \mathbb{L}^2(\Omega)^{N \times N}. \end{cases}$$

**(21)**
$$\begin{cases} \phi : \Omega \times S_N \times S_N \times \mathbb{R}^M \to \mathbb{R}^M \text{ and} \\ (\mathbf{a}) \exists C > 0 \text{ that } \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, \ k_1, k_2 \in \mathbb{R}^M \text{ a.e in } \Omega \\ |\phi(x, \sigma_1, \varepsilon_1, k_1) - \phi(x, \sigma_2, \varepsilon_2, k_2)| \leq C(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |k_1 - k_2|) \\ (\mathbf{b}) \quad x \to \phi(x, \sigma, \varepsilon, k) \text{ is mesurabl function with respect to the Lebesgue measure on } \Omega \text{ for all } \sigma, \varepsilon \in S_N, \ k \in \mathbb{R}^M \\ (\mathbf{c}) x \to \phi(x, 0, 0, 0) \in \mathbb{L}^2(\Omega)^M. \end{cases}$$

$\Phi : \Gamma_2 \to S_N$ such that :

$$\textbf{(22)} \quad \begin{cases} \Phi(x)u.u \geq 0 \ \text{ on } \Gamma_2. \\ \Phi_{km} \in L^\infty(\overline{\Gamma}_2) \ \text{ pour } k,m = 1,...,N \end{cases}$$

$$\textbf{(23)} \quad k_0 \in \mathbb{L}^2(\Omega)^M$$

The main result of this section is the following:

**Theorem 1 (1)** *Assume (8), (9), (14), (19) and (20)-(23), the problem (2) have a unique solution*

$$\textbf{(24)} \quad u \in W^{1,\infty}(0,T,V_1),\ \sigma \in W^{1,\infty}(0,T,Q_1), k \in W^{1,\infty}(0,T,\mathbb{L}^2(\Omega)^M).$$

**Proof:** We start by the existence part.
Let $X$ be the product Hilbert space $X = Q \times \mathbb{L}^2(\Omega)^M$ and let $\eta = (\eta^1, \eta^2)$ in $C^0(0,T,X)$.
We define the function $Z_\eta = Z_\eta^1 + Z_\eta^2 \in C^0(0,T,X)$ by

$$\textbf{(25)} \qquad Z_\eta(t) = \int_0^t \eta(s)ds + Z_0 \ \text{ where}$$

$$\textbf{(26)} \qquad Z_0 = (\sigma_0 - \xi\varepsilon(u_0), k_0)$$

Consider the following auxiliary variational problem
**Problem 2** Find a displacement field $u_\eta : [0,T] \to \mathbb{V}_1$, a stress field $\sigma_\eta : [0,T] \to Q_1$and $k \in W^{1,\infty}(0,T,\mathbb{L}^2(\Omega)^M)$

$$\textbf{(27)} \qquad \sigma_\eta(t) = \xi(\varepsilon(u_\eta)) + Z_\eta^1(t) \quad \text{in } \Omega \times (0,T)$$

$$\textbf{(28)} \qquad \langle \sigma_\eta(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)) \rangle_Q + \langle \Phi u(t), v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} + j(v) - j(\dot{u}_\eta(t)) \geq \langle f(t), v - \dot{u}_\eta(t) \rangle_V \qquad \forall v \in V_1$$

$$\textbf{(29)} \qquad u_\eta(0) = u_0,\ \sigma_\eta(0) = \sigma_0,\ k(0) = k_0.$$

**Lemma 2.** *The variational problem (2) has a unique solution* $u_\eta \in W^{1,\infty}(0,T,V_1)$, $\sigma_\eta \in W^{1,\infty}(0,T,Q_1)$.
**Proof:** Define a bilinear form $a : V_1 \times V_1 \to \mathbb{R}$ by

$$a(u,v) = \langle \xi(\varepsilon(u)), \varepsilon(v) \rangle_Q + \langle \Phi u(t), v - \dot{u}(t) \rangle_{\mathbb{L}^2(\Gamma_2)^N} \quad \forall u, v \in V_1$$

it follows from the assumptions (19) on $\xi$ that $a(u,v)$ is continuous, symetric

and $V_1$-elliptic. Also by (9) we see $j$ is a continuous seminorm on $V_1$.
Let $f_{_\eta} : [0,T] \to V_1$ by

$$\left\langle f_{_\eta}, v\right\rangle_V = \left\langle f(t), v\right\rangle_V - \left\langle Z^1_\eta(t), \varepsilon(v)\right\rangle_Q \quad \forall v \in V_1.$$

Since $f \in W^{1,\infty}([0,T], V_1)$ and $Z^1_\eta \in W^{1,\infty}([0,T], Q)$, we have $f_{_\eta} \in W^{1,\infty}([0,T], V_1)$. Moreover, since $\sigma_0 = \xi\varepsilon(u_0) + Z^1_0 = \xi\varepsilon(u_0) + Z^1_{_\eta}(0)$ by (26) and (25), the assumtion $\sigma_0 \in \Sigma(0)$ from (14) can be written as

$$\left\langle \xi\varepsilon(u_0), \varepsilon(v)\right\rangle_Q + \left\langle Z^1_{_\eta}(0), \varepsilon(v)\right\rangle_Q + j(v) \geq \left\langle f(0), v\right\rangle_V \qquad \forall v \in V_1$$

Applying a theorem in ([3] page70) we see that there exists a unique solution $u_{_\eta} \in W^{1,\infty}(0,T,V_1)$ to the problem

**(30)** $$a(u_\eta(t), v - \dot{u}_\eta(t)) + j(v) - j(\dot{u}_\eta(t)) \geq \left\langle f_{_\eta}(t), v - \dot{u}_\eta(t)\right\rangle_V \qquad \forall v \in V_1$$

**(31)** $$u_\eta(0) = u_0$$

Let $\sigma_{_\eta} \in W^{1,\infty}(0,T,Q)$ be given by (26). Then $(u_\eta, \sigma_{_\eta})$ is a solution for (27)-(29). From inequality (28) we obtain $Div\sigma_\eta + f_0 = 0$ a.e in $\Omega \times (0,T)$ . Therefore, $\sigma_{_\eta} \in W^{1,\infty}(0,T,Q_1)$.
Let $k_\eta \in L^\infty(0,T,\mathbb{L}^2(\Omega)^M)$ be the function defined by

**(32)** $$k_\eta = Z^2_\eta$$

Finaly, the uniqueness part of the lemma follows from the unique solvability of the variational inequality problem (30)-(31).
We consider next an operator $\Lambda : L^\infty(0,T,X) \to L^\infty(0,T,X)$ defined by

**(33)** $\Lambda\eta(t) = (G(\sigma_\eta(t),\, \varepsilon(u_{_\eta}),\, k_\eta(t)), \phi(\sigma_\eta(t),\, \varepsilon(u_{_\eta}),\, k_\eta(t)))$ for all $t \in [0,T]$.

Where $(u_\eta, \sigma_{_\eta}, k_{_\eta})$ is solution of problem (3).
**Lemma 3**. *The operator* $\Lambda$ *has a unique fixed point* $\eta_* \in L^\infty(0,T,X)$.
**Proof.**Let $\eta_1,\, \eta_2 \in L^\infty(0,T,X)$ and denote
$u_i = u_{\eta i_i},\, \sigma_{i_i} = \sigma_{\eta i_i},\, Z_{i_i} = Z_{\eta i_i},\, Z_i = Z^1_i + Z^2_i$
and $\eta_i = (\eta^1_i, \eta^2_i)$ for all $i = 1,2$. Rewrite (27) and (28) for $u_1$ and $u_2$ as

$$a(u_1(t), v - \dot{u}_1(t)) + \left\langle Z^1_{_1}, \varepsilon(v) - \varepsilon(\dot{u}_1(t))\right\rangle_Q + j(v) - j(\dot{u}_1(t)) \geq \left\langle f(t), v - \dot{u}_1(t)\right\rangle_V$$

$$a(u_2(t), v - \dot{u}_2(t)) + \left\langle Z^1_2, \varepsilon(v) - \varepsilon(\dot{u}_2(t))\right\rangle_Q + j(v) - j(\dot{u}_2(t)) \geq \left\langle f(t), v - \dot{u}_2(t)\right\rangle_V$$

For all $v \in V_1$, a.e on $(0,T)$. we take $v = \dot{u}_1$ in the first inequality, $v = \dot{u}_2$ in the second inequality, and add the two inequalities to obtain

$$a(u_2 - u_1, \dot{u}_2 - \dot{u}_1) \leq -\left\langle Z_1^1 - Z_2^1, \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t))\right\rangle_Q \text{ a.e. on } (0,T).$$

Let $t \in \overline{[0,T]}$. Integrate the inequality from 0 to t:

$$\frac{1}{2}\frac{d}{dt}a(u_2 - u_1, u_2 - u_1) \leq -\left\langle Z_1^1 - Z_2^1, \varepsilon(u_1(t)) - \varepsilon(u_2(t))\right\rangle_Q +$$

$$+\int_0^t \left\langle \eta_1^1(s) - \eta_2^1(s), \varepsilon(u_1(s)) - \varepsilon(u_2(s))\right\rangle_Q ds$$

Then,

$$c\left\|u_1(t) - u_2(t)\right\|_V^2 \leq \left\|Z_1^1(t) - Z_2^1(t)\right\|_Q \left\|u_1(t) - u_2(t)\right\|_V +$$

$$+\int_0^t \left\|\eta_1^1(s) - \eta_2^1(s)\right\|_Q \left\|u_1(s) - u_2(s)\right\|_V ds.$$

So

$$\left\|Z_1^1(t) - Z_2^1(t)\right\|_Q \leq \int_0^t \left\|\eta_1^1(s) - \eta_2^1(s)\right\|_Q ds.$$

Therefre, we have

$$\left\|u_1(t) - u_2(t)\right\|_V^2 \leq c\int_0^t \left\|\eta_1^1(s) - \eta_2^1(s)\right\|_Q^2 ds + \int_0^t \left\|u_1(s) - u_2(s)\right\|_V^2 ds.$$

Applying the Gronwal inequality, we obtain

**(34)** $\left\|u_1(t) - u_2(t)\right\|_V^2 \leq c\int_0^t \left\|\eta_1^1(s) - \eta_2^1(s)\right\|_Q^2 ds$

By the definition (27) and (33) we have

**(35)** $\left\|\sigma_1(t) - \sigma_2(t)\right\|_V^2 \leq c\int_0^t \left\|\eta_1^1(s) - \eta_2^1(s)\right\|_Q^2 ds$

Finally,

$$\begin{aligned}\Lambda\eta_1(t) - \Lambda\eta_2(t) &= (G(\sigma_{\eta_1}(t), \varepsilon(u_{\eta_1}), K_{\eta_1}(t)), \phi(\sigma_{\eta_1}(t), \varepsilon(u_{\eta_1}), K_{\eta_1}(t))) - \\ &\quad (G(\sigma_{\eta_2}(t), \varepsilon(u_{\eta_2}), K_{\eta_2}(t)), \phi(\sigma_{\eta_2}(t), \varepsilon(u_{\eta_2}), K_{\eta_2}(t)))\end{aligned}$$

Using now (20), (21) , (34) and (33) we get

**(36)** $|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_X \leq \tilde{C} \int_0^t |\eta_1(s) - \eta_2(s)|_X \, ds$ for all $t \in (0,T)$

Lemma (3) follows from (36) and an application of banach's fixed point theorem.
Now we are ready to prove Theorem (1).
**Proof of theorem (1)**
**Existence.**
Let $\eta^* \in L^\infty(0,T,X)$ be the fixed point of $\Lambda$ and let $(u_{\eta^*}, \sigma_{\eta^*}, K_{\eta^*})$ be the solution of problem (3) for $\eta = \eta^*$. Using (27) and (25) we have

$$\dot{\sigma}_{\eta^*}(t) = \xi(\varepsilon(\dot{u}_{\eta^*}(t))) + \eta^*(t) \quad a.e. \quad t \in (0,T).$$

Since

$$\eta^*(t) = \Lambda\eta^*(t) = (G(\sigma_\eta(t), \varepsilon(u_\eta), K_\eta(t)), \phi(\sigma_\eta(t), \varepsilon(u_\eta), K_\eta(t)))a.et \in (0,T).$$

it follows that $(u_{\eta^*}, \sigma_{\eta^*}, K_{\eta^*})$ satisfiers (16). Using now (28) and (29) we coclude that is a solution of problem (2) with rigularity (24).
**Uniqueness.** The uniquencess part follows from the uniquencess of the fixed point of the operator $\Lambda$.

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