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An Inequality for the Psi Functions[1]

Lingli Wu

School of Educational Science and Technology
Huzhou Teachers College, Huzhou 313000, P. R. China

Yuming Chu[2]

Department of Mathematics
Huzhou Teachers College, Huzhou 313000, P. R. China
chuyuming@hutc.zj.cn

Abstract

For $x > 0$, let $\Gamma(x)$ be the Euler's gamma function, and $\psi(x) = \Gamma'(x)/\Gamma(x)$ the psi function. In this paper, we prove that $|\psi^{(n)}(b) - \psi^{(n)}(a)| < (b - L(a,b))|\psi^{(n+1)}(b)| + (L(a,b) - a)|\psi^{(n+1)}(a)|$ for all $b > a > 0$ and $n = 0, 1, 2, \cdots$, where $L(a,b) = (b-a)/(\log b - \log a)$.

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1. Introduction

For real and positive values of x, the Euler's gamma function and its logarithmic derivative ψ, the so-called psi function, are defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.1}$$

For extensions of these functions to complex variables and for basic properties see [19].

Over the past half century many authors have established inequalities for these important functions (see [1-5,7-9,11,13,14,16,18] and the references therein). It was shown in [10,12] that gamma and psi functions inequalities have interesting applications in the theory of 0-1 matrices and in graph theory.

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[2]Corresponding author

For $b > a > 0$, the generalized logarithmic mean $L_p(a,b)$ of a and b is defined as

$$L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, & p \neq -1, 0, \\ \frac{b-a}{\log b - \log a}, & p = -1, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases}$$

It is well-known that $L_p(a,b)$ is a increasing function on p for fixed a and b. Denote $A(a,b) = L_1(a,b) = (b+a)/2$, $I(a,b) = L_0(a,b) = \left(b^b/a^a\right)^{1/(b-a)}/e$, $L(a,b) = L_{-1}(a,b) = (b-a)/(\log b - \log a)$, $G(a,b) = L_{-2}(a,b) = \sqrt{ab}$ are the arithmetic mean, identric mean, logarithmic mean and geometric mean of a and b, respectively.

Recently, N. Batir [6] proved

$$|\psi^{(n)}(b) - \psi^{(n)}(a)| < (b-a)|\psi^{(n+1)}(L_{-(n+2)}(a,b))|, \quad n = 1, 2, 3, \cdots. \tag{1.2}$$

The purpose of this paper is to establish the following new upper bound for $|\psi^{(n)}(b) - \psi^{(n)}(a)|$:

Theorem. *If $b > a > 0$, $n = 0, 1, 2, \cdots$, then*

$$\left|\psi^{(n)}(b) - \psi^{(n)}(a)\right| < (b - L(a,b))\left|\psi^{(n+1)}(b)\right| + (L(a,b) - a)\left|\psi^{(n+1)}(a)\right|.$$

2. Lemmas and Proof of Theorem

First we introduce three definitions:

Definition 2.1. Let $I \subseteq R$ be an interval, $f : I \to R$ is called a convex(concave) function if $f(\alpha x + (1-\alpha)y) \leq (\geq) \alpha f(x) + (1-\alpha) f(y)$ for all $x, y \in I$ and $\alpha \in [0,1]$.

Definition 2.2. Let $I \subseteq R$ be an interval, $f : I \to (0, +\infty)$ is called a logarithmically convex(concave) function if $f(\alpha x + (1-\alpha)y) \leq (\geq) f(x)^\alpha f(y)^{1-\alpha}$ for all $x, y \in I$ and $\alpha \in [0,1]$.

Definition 2.3. Let $I \subseteq (0, +\infty)$ be an interval, $f : I \to R$ is called a $GA-$convex(concave) function if $f(x^\alpha y^{1-\alpha}) \leq (\geq) \alpha f(x) + (1-\alpha) f(y)$ for all $x, y \in I$ and $\alpha \in [0,1]$.

Next we shall establish and introduce the following five lemmas, they are the key of the proof of our main result in this section.

Lemma 2.1. *Let $b > a > 0$, $c \in (a,b)$. If $f : [a,b] \to R$ is a differentiable GA-convex(concave) function, then*

$$\begin{aligned} &(b-a)f(c) + cf'(c)((\log b - \log c)(b - L(c,b)) - (\log c - \log a)(L(a,c) - a)) \\ &\quad \leq (\geq) \int_a^b f(x)dx \leq (\geq)(b - L(a,b))f(b) + (L(a,b) - a)f(a), \end{aligned} \tag{2.1}$$

with equality(for left or right hand side) if and only if $f(x) = p\log x + q$ *for some* $p, q \in R$.

Proof. To prove the left hand side inequality in (2.1). For $x \in [c, b]$, let $c_1 = \log c$ and $x_1 = \log x$. Taking $g(t) = f(e^t)$, then $g : [\log a, \log b] \to R$ is a convex(concave) function because of f is a GA-convex(concave) function. This and the Lagrange mean value theorem yield

$$\frac{g(x_1) - g(c_1)}{x_1 - c_1} \geq (\leq) g'(c_1), \tag{2.2}$$

this leads to

$$f(x) - f(c) \geq (\leq) cf'(c)(\log x - \log c). \tag{2.3}$$

Next let $h(x) = \int_c^x f(t)dt - (x-c)f(c) - cf'(c)(x(\log x - \log c) - x + c)$, $x \in [c, b]$. Then $h(c) = 0$ and $h'(x) \geq (\leq) 0$ for $x \in [c, b]$ by (2.3), this implies $h(x) \geq (\leq) 0$ for all $x \in [c, b]$. Hence $h(b) \geq (\leq) 0$, this yields

$$\begin{aligned} \int_c^b f(x)dx &\geq (\leq)(b-c)f(c) + cf'(c)(b(\log b - \log c) - b + c) \\ &= (b-c)f(c) + cf'(c)(\log b - \log c)(b - L(c, b)). \end{aligned} \tag{2.4}$$

The similar argument as above gives

$$\int_a^c f(x)dx \geq (\leq)(c-a)f(c) + cf'(c)(\log c - \log a)(a - L(a, c)). \tag{2.5}$$

Combining (2.4) and (2.5) we can get the left hand side inequality in (2.1).

To prove the right hand side inequality in (2.1). For any $x \in [a, b]$, taking $y = (\log x - \log a)/(\log b - \log a)$. Then $0 \leq y \leq 1$ and $x = a^{1-y}b^y$, by the definition of GA-convex(concave) function and the transformation to variable of integration, we have

$$\begin{aligned} \int_a^b f(x)dx &= \int_0^1 f(a^{1-y}b^y)d(a^{1-y}b^y) \\ &\leq (\geq) \ a\int_0^1 ((1-y)f(a) + yf(b))d(\frac{b}{a})^y \\ &= bf(b) - af(a) - L(a, b)(f(b) - f(a)) \\ &= (b - L(a, b))f(b) + (L(a, b) - a)f(a). \end{aligned} \tag{2.6}$$

At last, from the above argument, it is easy to see that the left or right hand side inequality becomes equality if and only if $f(e^x) = px + q$ for some $p, q \in R$, namely, $f(x) = p\log x + q$.

Lemma 2.2.(see[17]) *Let* $I \subseteq (0, +\infty)$ *be an interval. If* $f : I \to R$ *is a twice differentiable function, then* f *is a* GA*-convex(concave) function in* I *if and only if* $xf'(x) + x^2 f''(x) \geq (\leq) 0$ *for all* $x \in I$.

Lemma 2.3.(see[11,20]) *For any $x > 0$, the following statements are true:*

$$(a) \quad 2\psi'(x) + x\psi''(x) < \frac{1}{x}, \tag{2.7}$$

$$(b) \quad \psi'(x) > \frac{1}{x} + \frac{1}{2x^2}. \tag{2.8}$$

Lemma 2.4.(see [15]) *Let $x > 0$, $n = 0, 1, 2, \cdots$. If $0 \le \alpha \le n$, then*

$$x|\psi^{(n+1)}(x)| - \alpha|\psi^{(n)}(x)| > 0. \tag{2.9}$$

Lemma 2.5. *If $b > a > 0$, $n = 0, 1, 2, \cdots$, then $(-1)^n\psi^{(n)}(x)$ is a GA-concave function in $[a, b]$.*
Proof. If $n = 0$, then (2.7) and (2.8) lead to

$$\begin{aligned} & x\psi'(x) + x^2\psi''(x) \\ < \ & x(\frac{1}{x} - \psi'(x)) \\ < \ & -\frac{1}{2x} < 0. \end{aligned}$$

This and Lemma 2.2 imply that $\psi(x)$ is a GA-concave function.

Next we assume that $n \ge 1$. It is well-known that $\log\Gamma(x) = -cx + \sum_{k=1}^{\infty}((x/k) - \log(1 + (x/k))) - \log x$, where $c = 0.577215\cdots$ is the Euler's constant. From this we can get

$$\psi^{(n)}(x) = (-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(k+x)^{n+1}}. \tag{2.10}$$

(2.10) and Lemma 2.4 lead to

$$\begin{aligned} & x((-1)^n\psi^{(n)}(x))' + x^2((-1)^n\psi^{(n)}(x))'' \\ = \ & (n+1)!x\sum_{k=0}^{\infty}\frac{1}{(k+x)^{n+2}} - (n+2)!x^2\sum_{k=0}^{\infty}\frac{1}{(k+x)^{n+3}} \\ = \ & -x\left(x\left|\psi^{(n+2)}(x)\right| - \left|\psi^{(n+1)}(x)\right|\right) \\ < \ & 0. \end{aligned} \tag{2.11}$$

(2.11) and Lemma 2 imply that $(-1)^n\psi^{(n)}(x)$ is a GA-concave function in $[a, b]$.

Now we can prove our Theorem.

Proof of Theorem. By Lemma 2.5 we know that $(-1)^n\psi^{(n+1)}(x)$ is a GA-convex function in $[a, b]$, making use of Lemma 2.1 and (2.10) we get

$$\left|\psi^{(n)}(b) - \psi^{(n)}(a)\right|$$

$$
\begin{aligned}
&= \left| \int_a^b \psi^{(n+1)}(x)dx \right| \\
&= \int_a^b (-1)^n \psi^{(n+1)}(x)dx \\
&< (b - L(a,b))(-1)^n \psi^{(n+1)}(b) + (L(a,b) - a)(-1)^n \psi^{(n+1)}(a) \\
&= (b - L(a,b)) \left| \psi^{(n+1)}(b) \right| + (L(a,b) - a) \left| \psi^{(n+1)}(a) \right| .
\end{aligned}
$$

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