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Viscosity Approximation Methods for Mean Non-Expansive Mappings in Banach Spaces

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Abstract. Let C be a nonempty closed convex subset of a real reflexive Banach space X that has weakly continuous duality mapping J. Let $T : C \to C$ be a Mean non-expansive mapping with $F(T) \neq \emptyset$. For any $t \in (0, 1)$, there exists a sequence $\{x_t\} \in C$ satisfying $x_t = tf(x_t)+(1-t)Tx_t$, where $f : C \to C$ is a contraction mapping. Then it is proved that $\{x_t\} \in C$ converges strongly to a fixed point of T which is also a solution of certain variational inequality.

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1. Introduction

Let X be a real Banach space, and let J denote the normalized duality from X into 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\},\ \forall x \in X,$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in X : Tx = x\}$. When $\{x_n\}$ is a sequence in X, then $x_n \to x(x_n \rightharpoonup x, x_n \rightharpoondown x)$ will denote strong(weak, weak star)convergence of the sequence$\{x_n\}$ to x.

Let X be a real Banach space and T a mapping with domain $D(T)$ and range $R(T)$ in X. T is called non-expansive(contractive) if for any $x, y \in D(T)$ such that

$$\|Tx - Ty\| \leq \|x - y\|(\|Tx - Ty\| \leq \alpha\|x - y\|\ for\ some\ 0 < \alpha < 1).$$

In 1967, Browder [2] considered an iteration in a Hilbert space as follows. Let u be an arbitrary point of C and define a contraction by

$$T_t{}^f : x \mapsto tu + (1-t)Tx, \ x \in C, \tag{1}$$

where $t \in (0,1)$. it proved that the fixed point sequence $\{x_t\}$ of $\{T_t{}^f\}$ converges as $t \to 0$ strongly to a fixed point of T. In 1980, Reich [7] extended the result of Browder to a uniformly smooth Banach spaces.

In 2000, Moudafi [6] introduced viscosity approximation methods and proved that if X is a real Hilbert space, the sequence $\{x_t\}$ defined by the following:

$$x_t = tf(x_t) + (1-t)Tx_t, \tag{2}$$

converges strongly to a fixed point of the non-expansive self-mapping T in C which is the unique solution to the following variational inequality:

$$\langle (I-f)u^*, J(p-u^*)\rangle \geq 0, \ \forall \in F(T).$$

In 2004, Xu [9] studied further the viscosity approximation methods for non-expansive mappings in uniformly smooth Banach space, and proved that as $t \downarrow 0$, $\{x_t\}$ defined by (2) converges to a point in $F(T)$ that is the unique solution of the variation inequality.

A Banach space X is said to admit a weakly sequentially continuous normalized duality mapping $J : X \to X^*$, if J is single-value and weak-weak* continuous, i.e., for any sequence$\{x_n\}$ in X, if $x_n \rightharpoonup x$ in X, then $J(x_n) \rightharpoondown J(x)$ in X^*.

In 2006, Xu [10] proved the strong convergence of $\{x_t\}$ defined by (1) in a reflexive Banach space with a weakly continuous duality map J_φ with gauge φ. And it also considered the following iterative scheme:

$$x_{n+1} = (1-\alpha_n)J_{r_n}x_n + \alpha_n u, \ n \geq 0, \tag{3}$$

where $u \in C$ is arbitrarily fixed, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{r_n\}$ is a sequence of positive numbers. Xu proved that if X is a reflexive Banach space with weakly continuous duality mapping, then the sequence $\{x_n\}$ given by (3) converges strongly to a point in $F(T)$ provided the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy certain conditions.

Let X be a real Banach space, C a bounded closed convex subset of X and $T : C \to C$ be a mapping, T is called a mean non-expansive mapping if

$$\|Tx - Ty\| \leq a\|x-y\| + b\|x - Ty\|, \ \forall x, y \in C, a, b \geq 0, a + b \leq 1. \tag{4}$$

In 1975, Zhang [12] introduced this definition and proved that T has a unique fixed point in C, where C is a weakly compact closed convex subset and has normal structure. In 2007, Wu [8] proved that If $a+b<1$, then mean non-expansive T defined by (4) has a unique fixed point.

The objective of this paper is to consider the following two iterations for a mean non-expansive mapping T in a reflexive Banach space X which has a weakly continuous duality mapping:

$$x_t = tf(x_t) + (1-t)Tx_t,\ t \in (0,1),$$
$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n),\ n \geq 0. \tag{5}$$

2. PRELIMINARIES

To this purpose, let us first recall the following some lemmas.

Lemma 1 [3] *If X is a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping, then X satisfies the Opial's condition, i.e., wheneven $x_n \rightharpoonup x$ in X and $y \neq x$, then*

$$\liminf \|x_n - x\| < \liminf \|x_n - y\|.$$

Lemma 2 [5] *Let X be a real Banach space. For each $x, y \in X$, the following conclusions hold:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y)\ ,$$
$$\|x+y\|^2 \geq \|x\|^2 + 2\langle y, j(x)\rangle, \quad \forall j(x) \in J(x)\ .$$

Lemma 3 [4] *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n$$

with $\{t_n\} \subset [0,1]$, $\sum\limits_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum\limits_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$.

Lemma 4 *Let X be a Banach space with a weakly sequentially continuous normalized duality mapping, C a bounded closed convex subset of X and let $\{x_n\}$ be a bounded sequence of X and $u \in C$. Then*

$$LIM\|x_n - u\|^2 = \min_{y \in C} LIM\|x_n - y\|^2$$

if and only if

$$LIM\,\langle z - u, j(x_n - u)\rangle \leq 0$$

for all $z \in C$, where LIM is a Banach limit on ℓ^∞.

Proof. For z in C and $\lambda : 0 \leq \lambda \leq 1$, we have by Lemma 2 that

$$\begin{aligned}\|x_n - u\|^2 &= \|x_n - \lambda u - (1-\lambda)z + (1-\lambda)(z-u)\|^2 \\ &\geq \|x_n - \lambda u - (1-\lambda)z\|^2 \\ &\quad +2(1-\lambda)\langle z-u, J(x_n - \lambda u - (1-\lambda)z)\rangle\end{aligned}$$

Let $\varepsilon > 0$ be given. Since X is reflexive which admits a weakly sequentially continuous duality mapping. Therefore,

$$|\langle z-u, J(x_n-\lambda u-(1-\lambda)z)-J(x_n-u)\rangle| < \varepsilon,$$

if λ is close enough to 1. Consequently, we have

$$\begin{aligned} |\langle z-u, J(x_n-u)\rangle| &< \varepsilon+\langle z-u, J(x_n-\lambda u-(1-\lambda)z)\rangle \\ &\leq \varepsilon+\frac{1}{2(1-\lambda)}\{\|x_n-u\|^2-\|x_n-\lambda u-(1-\lambda)z\|^2\} \end{aligned}$$

and hence

$$\begin{aligned} &LIM\langle z-u, J(x_n-u)\rangle \\ &\leq \varepsilon+\tfrac{1}{2(1-\lambda)}\{LIM\|x_n-u\|^2-LIM\|x_n-\lambda u-(1-\lambda)z\|^2\} < \varepsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitary, we have $LIM\ \langle z-u, j(x_n-u)\rangle \leq 0$ for all $z \in C$.

We prove the converse. Let z, $u \in C$. Then, by Lemma 2,

$$\|x_n-z\|^2-\|x_n-u\|^2 \geq 2\langle u-z, J(x_n-u)\rangle,$$

for all $n \geq 1$ and $LIM\langle z-u, J(x_n-u)\rangle \leq 0$,we have

$$LIM\|x_n-z\|^2 = \min_{x\in K} LIM\|x_n-x\|^2.$$

□

Remark 1. If we suppose that X be a Banach space with a uniformly Gateaux differentiable norm, then the duality map is uniformly continuous on bounded subset of X from the strong topology of X to the weak star topology of X^*(see [11]). Thus, it also satisfies the above result.

3. Main results

Let X be a Banach space, C a closed convex subset of X, $T : C \to C$ a mean non-expansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with contraction constant α. For given $t \in (0,1)$ define a napping $T_t : C \to C$ by

$$T_t(x) = tf(x)+(1-t)Tx,\ x \in C.$$

Clearly, for each $x_t \in C$, we have that T_t is mean non-expansive. Therefore, by Lemma 2.1 of [8], T_t has a unique fixed point(say) $x_t \in C$, that is

$$x_t = tf(x_t)+(1-t)Tx_t. \tag{6}$$

Concerning the convergence problem of sequence $\{x_t\}$, we can prove the following results.

Theorem 1 *Let X be a real reflexive Banach space with a weakly sequentially continuous normalized duality mapping $J : X \to X^*$, C a closed convex subset of X, $T : C \to C$ defined by (4) a mean non-expansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ be a contraction with contraction constant α. Then $\{x_t\}$ defined by (6) converges strongly to a point in $F(T)$. If we define $Q : \prod_C \to F(T)$ by*

$$Q(f) := \lim_{t \to 0} x_t,$$

where $\prod_C := \{f : C \to C$ contraction with contraction constant $\alpha\}$, then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \ \ p \in F(T) \tag{7}$$

Proof. We first show that the sequence $\{x_t\}$ defined by (6) is bounded. In fact, take a $p \in F(T)$, we have

$$\begin{aligned} \|x_t - p\| &\leq (1-t)\|Tx_t - p\| + t\|f(x_t) - p\| \\ &\leq (1-t)(a\|x_t - p\| + b\|x_t - Tp\|) + t\|f(x_t) - p\| \\ &= (1-t)(a\|x_t - p\| + b\|x_t - p\|) + t\|f(x_t) - p\| \\ &\leq (1-t)\|x_t - p\| + t\|f(x_t) - p\| \end{aligned}$$

It follows that

$$\begin{aligned} \|x_t - p\| &\leq \|f(x_t) - p\| \\ &\leq \|f(x_t) - f(p)\| + \|f(p) - p\| \\ &\leq \alpha\|x_t - p\| + \|f(p) - p\| \end{aligned}$$

Hence

$$\|x_t - p\| \leq \frac{1}{1-\alpha}\|f(p) - p\| \tag{8}$$

and $\{x_t\}$ is bounded. Assume $t_n \to 0$. Let $x_n := x_{t_n}$, then $\{x_n\}$ is bounded, so are $\{fx_n\}$. We claim that

$$\|x_n - Tx_n\| \to 0 \tag{9}$$

Since

$$\begin{aligned} \|x_n - Tx_n\| &= \|t_n f(x_n) + (1-t_n)Tx_n - Tx_n\| \\ &= t_n\|f(x_n) - Tx_n\| \\ &= t_n\|f(x_n) - p + p - Tx_n\| \\ &\leq t_n(\|f(x_n) - p\| + \|Tx_n - p\|) \\ &\leq t_n(\|f(x_n) - p\| + a\|x_n - p\| + b\|x_n - Tp\|) \\ &\leq t_n(\|f(x_n) - p\| + \|x_n - p\|) \end{aligned}$$

Let $M \geq 2max\{\|f(x_n) - p\|, \|x_n - p\|\}$, we have

$$\|x_n - Tx_n\| \leq t_n M \to 0 (as\ t_n \to 0).$$

Now we define $\mu : C \to \mathbb{R}$ by

$$\mu(x) = LIM\|x_n - x\|^2, \ x \in C,$$

Let

$$K = \{x \in C : \mu(x) = \min_{x \in C} LIM\|x_n - x\|^2\}.$$

It is easily seen that K is a nonempty closed convex bounded subset of X. Since (note that $\|x_n - Tx_n\| \to 0$)

$$\mu(Tx) = LIM\|x_n - Tx\|^2 = LIM\|Tx_n - Tx\|^2,$$

and

$$\begin{aligned} LIM\|Tx_n - Tx\|^2 &\leq LIM(a\|x_n - x\| + b\|x_n - Tx\|)^2 \\ &= LIM(a^2\|x_n - x\|^2 + b^2\|x_n - Tx\|^2 \\ &\quad +2ab\|x_n - x\|\|x_n - Tx\|) \\ &\leq LIM(a^2\|x_n - x\|^2 + 2ab\|x_n - Tx\|^2 + b^2\|x_n - Tx\|^2) \end{aligned}$$

Hence

$$\mu(Tx) \leq \frac{a^2}{1 - b^2 - 2ab} LIM\|x_n - x\|^2 \leq LIM\|x_n - x\|^2 = \mu(x),$$

it follows that $T(K) \subset K$, that is, K is invariant under T. since X is reflexive, we get that μ attains its infimum over K(see [1]). That is there exists a $y \in K$ such that

$$LIM\|x_n - y\|^2 = \min_{x \in C} LIM\|x_n - x\|^2$$

We next proved that $y = T(y)$. Suppose, by way of contradiction, that $y \neq T(y)$. Since $\{x_n\}$ is bounded, without lose of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$, then

$$\begin{aligned} LIMinf\|x_n - T(y)\|^2 &\leq LIMinf\|Tx_n - T(y)\|^2 \\ &\leq LIMinf(a\|x_n - y\| + b\|x_n - T(y)\|)^2 \\ &\leq LIMinf(a^2\|x_n - y\|^2 + 2ab\|x_n - T(y)\|^2 \\ &\quad +b^2\|x_n - T(y)\|^2) \end{aligned}$$

Hence

$$\begin{aligned} LIMinf\|x_n - T(y)\| &\leq \frac{a}{\sqrt{1 - b^2 - 2ab}} LIMinf\|x_n - y\| \\ &\leq LIMinf\|x_n - y\| \\ &\leq LIMinf\|x_n - x^*\| \end{aligned}$$

on the other hand, From Lemma 1 we get that

$$LIMinf\|x_n - x^*\| < LIMinf\|x_n - T(y)\|,$$

a contradiction. Thus $y = T(y)$. That is, y is a fixed point of T, we also have by Lemma 4 that

$$LIM\langle x - y, J(x_n - y)\rangle \le 0, \; x \in C \tag{10}$$

Since

$$\begin{aligned} \|x_t - y\|^2 &= \|t(f(x_t) - y) + (1-t)(Tx_t - y)\|^2 \\ &= \langle t(f(x_t) - y) + (1-t)(Tx_t - y), J(x_t - y)\rangle \\ &\le t\langle f(x_t) - y, J(x_t - y)\rangle + (1-t)\|Tx_t - y\|\|x_t - y\| \\ &\le t\langle f(x_t) - y, J(x_t - y)\rangle + (1-t)(a\|x_t - y\| \\ &\quad +b\|x_t - T(y)\|)\|x_t - y\| \\ &\le t\langle f(x_t) - y, J(x_t - y)\rangle + (1-t)\|x_t - y\|^2 \end{aligned}$$

then

$$\|x_t - y\|^2 \le \langle f(x_t) - y, J(x_t - y)\rangle$$

$$= \langle f(x_t) - x, J(x_t - y)\rangle + \langle x - y, J(x_t - y)\rangle \tag{11}$$

Hence, for all $x \in C$

$$\begin{aligned} LIM\|x_n - y\|^2 &\le LIM\langle f(x_n) - x, J(x_n - y)\rangle + LIM\langle x - y, J(x_n - y)\rangle \\ &\le LIM\langle f(x_n) - x, J(x_n - y)\rangle \\ &\le LIM\|f(x_n) - x\|\|x_n - y\| \end{aligned}$$

In particular, let $x = f(y)$,

$$LIM\|x_n - y\|^2 \le LIM\|f(x_n) - f(y)\|\|x_n - y\| \le \alpha LIM\|x_n - y\|^2$$

Hence,

$$LIM\|x_n - y\|^2 = 0,$$

and there exists a subsequence of $\{x_t\}$ which is still denoted $\{x_n\}$ such that $x_n \to y$.

Now assume there exists another subsequence $\{x_m\}$ of $\{x_t\}$ such that $x_m \to y^* \in F(T)$.
It follows from (11) that

$$\|y^* - y\|^2 \le \langle f(y^*) - y, J(y^* - y)\rangle$$

Interchange y^* and y to obtain

$$\|y - y^*\|^2 \le \langle f(y) - y^*, J(y - y^*)\rangle$$

Which implies that

$$2\|y^* - y\|^2 \le \langle f(y^*) - y, J(y^* - y)\rangle \le (1+\alpha)\|y^* - y\|^2.$$

Since $\alpha \in (0,1)$, this implies that $y^* = y$. Consequently, $x_t \to y$ as $t \to 0$. Now we show that $Q(f)$ satisfies (7).
Define $Q := \prod_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0} x_t. \tag{12}$$

We have by (6) that

$$(I-f)x_t = -\frac{1-t}{t}(I-T)x_t. \tag{13}$$

Hence for $p \in F(T)$,

$$\langle (I-f)x_t, J(x_t-p)\rangle = -\frac{1-t}{t}\langle (I-T)x_t-(I-T)p, J(x_t-p)\rangle \le 0. \tag{14}$$

Letting $t \to 0$, we claim that

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0 \tag{15}$$

In fact, let $x^{**} = Q(f)$ and since X is a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping. Hence

$$\begin{aligned} |\langle f(x^{**})-x^{**}, J(p-x^{**})\rangle &- \langle f(x_t)-x_t, J(p-x_t)\rangle| \\ &= |\langle f(x^{**})-x^{**}, J(p-x^{**})-J(p-x_t)\rangle \\ &\quad +\langle (f(x^{**})-x^{**})-(f(x_t)-x_t), J(p-x_t)\rangle| \\ &\le \langle f(x^{**})-x^{**}, J(p-x^{**})-J(p-x_t)\rangle \\ &\quad +\|(f(x^{**})-x^{**})-(f(x_t)-x_t)\|\|p-x_t\| \to 0 \end{aligned}$$

Thus, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\langle f(x^{**})-x^{**}, J(p-x^{**})\rangle < \langle f(x_t)-x_t, J(x_t-p)\rangle + \varepsilon \le \varepsilon,$$

for any $t \in (0,\delta)$ and for all $n \ge 1$. Since $\epsilon > 0$ is arbitrary, we have

$$\langle (I-f)x^{**}, J(x^{**}-p)\rangle \le 0.$$

□

Theorem 2 *Let X be a real reflexive Banach space with a weakly sequentially continuous normalized duality mapping $J : X \to X^*$, C a bounded closed convex subset of X, $T : C \to C$ defined by (4) a mean non-expansive mapping with $F(T) \neq \emptyset$ and $f \in \prod_C$. For any given $x_0 \in C$, let $\{x_n\}$ be the iterative sequence defined by (5) and $\{\alpha_n\}$ satisfies the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then $\{x_n\}$ convergence strongly to $Q(f)$, where $Q := \prod_C \to F(T)$ is defined by (12).

Proof. By (8), it is easy to prove that the sequence $\{x_n\}$ defined by (5) is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$.

We claim that

$$\|x_{n+1} - x_n\| \to 0. \tag{16}$$

Indeed we have(for some appropriate $M > 0$)

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)(Tx_n - Tx_{n-1}) + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - Tx_{n-1}) \\
&\quad +\alpha_n(f(x_n) - f(x_{n-1})\| \\
&\leq (1-\alpha_n)(a\|x_n - x_{n-1}\| + b\|x_n - Tx_{n-1}\|) \\
&\quad +|\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - Tx_{n-1}\| + \alpha\alpha_n\|x_n - x_{n-1}\| \\
&\leq (1-\alpha_n)(a\|x_n - x_{n-1}\| + \alpha_{n-1}\|f(x_{n-1}) - Tx_{n-1}\|) \\
&\quad +|\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - Tx_{n-1}\| + \alpha_n\|x_n - x_{n-1}\| \\
&\leq (1-\alpha_n)a\|x_n - x_{n-1}\| + ((1-\alpha_n)\alpha_{n-1} + |\alpha_n - \alpha_{n-1}|)M \\
&\quad +\alpha_n\|x_n - x_{n-1}\| \\
&= (1-(1-a)(1-\alpha_n)\|x_n - x_{n-1}\| + M|\alpha_n - \alpha_{n-1}| \\
&\quad +M(1-\alpha_n)\alpha_{n-1}.
\end{aligned}$$

If $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, then we can let $a_n = \|x_n - x_{n-1}\|$, $b_n = M(1-\alpha_n)\alpha_{n-1}$, $t_n = (1-a)(1-\alpha_n)$, $c_n = M|\alpha_n - \alpha_{n-1}|$, for any $n \geq 0$, then by Lemma 3, we have

$$\|x_{n+1} - x_n\| \to 0 (as\ n \to \infty).$$

If $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$, then let $c_n = 0$, $a_n = \|x_n - x_{n-1}\|$, $t_n = (1-a)(1-\alpha_n)$ and

$$b_n = \alpha_n \frac{M|\alpha_n - \alpha_{n-1}|}{\alpha_n} + M(1-\alpha_n)\alpha_{n-1} = |1 - \frac{\alpha_{n-1}}{\alpha_n}|\alpha_n M + M(1-\alpha_n)\alpha_{n-1},$$

for any $n \geq 0$, then the conditions of Lemma 3 are also satisfied. Hence, we have $\|x_{n+1} - x_n\| \to 0$.

We now show that

$$\|x_n - Tx_n\| \to 0.$$

Indeed this following from(16),

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| \to 0.
\end{aligned}$$

We next show that

$$\limsup_{n\to\infty} \langle x^* - f(x^*), J(x^* - x_n)\rangle \leq 0, \tag{17}$$

where $x^* = Q(f)$. Indeed we can write

$$x_t - x_n = t(f(x_t) - x_n) + (1-t)(Tx_t - x_n).$$

Putting

$$P_n(t) = \|Tx_n - x_n\|(2a\|x_t - x_n\| + 2b\|Tx_t - x_n\| + \|Tx_n - x_n\|) \to 0(n \to \infty)$$

and using Lemma 2, we obtain

$$\begin{aligned}\|x_t - x_n\|^2 &\leq (1-t)^2\|Tx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &\leq (1-t)^2\|Tx_t - x_n\|^2 + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle \\ &\quad +2t\|x_t - x_n\|^2.\end{aligned}$$

Since

$$\begin{aligned}\|Tx_t - x_n\|^2 &\leq (\|Tx_t - Tx_n\| + \|Tx_n - x_n\|)^2 \\ &\leq (a\|x_t - x_n\| + b\|Tx_t - x_n\| + \|Tx_n - x_n\|)^2 \\ &= a^2\|x_t - x_n\|^2 + b^2\|Tx_t - x_n\|^2 + 2ab\|x_t - x_n\|\|Tx_t - x_n\| \\ &\quad +\|Tx_n - x_n\|(2a\|x_t - x_n\| + 2b\|Tx_t - x_n\| + \|Tx_n - x_n\|) \\ &\leq a^2\|x_t - x_n\|^2 + b^2\|Tx_t - x_n\|^2 + ab(\|x_t - x_n\|^2 \\ &\quad +\|Tx_t - x_n\|^2) + P_n(t).\end{aligned}$$

Then

$$\begin{aligned}\|Tx_t - x_n\|^2 &\leq \frac{a^2 + ab}{1 - b^2 - ab}\|x_t - x_n\|^2 + \frac{P_n(t)}{1 - b^2 - ab} \\ &\leq \|x_t - x_n\|^2 + \frac{P_n(t)}{1 - b^2 - ab}.\end{aligned}$$

Hence

$$\begin{aligned}\|x_t - x_n\|^2 &\leq (1-t)^2\|x_t - x_n\|^2 + \frac{P_n(t)}{1 - b^2 - ab} \\ &\quad +2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2.\end{aligned}$$

This implies

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}P_n(t).$$

It follows that

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \leq M\frac{t}{2},$$

where $M > 0$ is a constant such that $M \geq \|x_t - x_n\|^2$ for all $n \geq 1$ and $t \in (0,1)$. Let $t \to 0$, then according to inequality (15) proved in Theorem 1, we obtain (17).

Now we let

$$\gamma_n = max\{\langle x^* - f(x^*), J(x^* - x_n)\rangle, 0\} \geq 0,$$

for any $n \geq 0$, we can prove that $\lim_{n\to\infty} \gamma_n = 0$. In fact, by (17), for given $\epsilon > 0$, there exists a natural $n_1 \in \mathbb{N}$ such that

$$\langle x^* - f(x^*), J(x^* - x_n)\rangle < \epsilon,$$

whenever $n \geq n_1$, thus, $0 \leq \gamma_n < \epsilon$, this implies $\lim_{n\to\infty} \gamma_n = 0$.

Finally we show that $x_n \to x^*$. Apply Lemma 2 to get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(Tx_n - x^*) + \alpha_n(f(x_n) - x^*)\|^2 \\
&\leq (1-\alpha_n)^2\|Tx_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), J(x_{n+1} - x^*)\rangle \\
&\quad +2\alpha_n\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha\alpha_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad +2\alpha_n\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha\alpha_n(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad +2\alpha_n\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha\alpha_n(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad +2\alpha_n\gamma_{n+1}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \frac{1-(2-\alpha)\alpha_n + {\alpha_n}^2}{1-\alpha\alpha_n}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha\alpha_n}\gamma_{n+1} \\
&\leq \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha\alpha_n}\gamma_{n+1} + M{\alpha_n}^2 \\
&= \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n - x^*\|^2 + \alpha_n(\frac{2}{1-\alpha\alpha_n}\gamma_{n+1} + M\alpha_n).
\end{aligned}$$

Let

$$a_n = \|x_n - x^*\|^2, t_n = \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}, b_n = \alpha_n(M\alpha_n + \frac{2}{1-\alpha\alpha_n}\gamma_{n+1}), c_n = 0,$$

then

$$\sum_{n=0}^{\infty} t_n = \sum_{n=0}^{\infty} \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n} > \sum_{n=0}^{\infty} 2\alpha_n(1-\alpha) = \infty,$$

and

$$\begin{aligned}
\frac{b_n}{t_n} &= \frac{1-\alpha\alpha_n}{2(1-\alpha)}(M\alpha_n + \frac{2}{1-\alpha\alpha_n}\gamma_{n+1}) \\
&\leq \frac{M\alpha_n}{2(1-\alpha)} + \frac{1}{1-\alpha}\gamma_{n+1} \to 0.
\end{aligned}$$

Finally apply Lemma 3 to conclude that $x_n \to x^*$. □

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