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A Study of a Two Variables

Gegenbauer Polynomials

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Abstract

The present paper deals with a study of a two variable polynomial $C_{n,k}^{\nu}(x,y)$ analogous to the Gegenbauer polynomial $C_n^{\nu}(x)$. The paper contains differential recurrence relations, a partial differential equation, double generating functions, double and triple hypergeometric forms, a special property and a bilinear double generating function for the newly defined polynomial $C_{n,k}^{\nu}(x,y)$.

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1. Introduction

The Hermite polynomials $H_n(x)$, the Legendre polynomials $P_n(x)$ and the Gegenbauer polynomial $C_n^{\nu}(x)$ are respectively defined by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}, \tag{1.1}$$

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1.2}$$

and

$$\left(1-2xt+t^2\right)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n \tag{1.3}$$

A careful inspection of the L.H.S. of (1.1), (1.2) and (1.3) reveals the fact that the L.H.S. of (1.1) is e^u, that of (1.2) is $(1 - u)^{-1/2}$ and that of (1.3) is $(1 - u)^{-\nu}$ where $u=2xt-t^2$. Thus $H_n(x)$, $P_n(x)$ and $C_n^{\nu}(x)$ are examples of polynomials generated by a function of the form $G(2xt - t^2)$. The expansions of $(1 - u)^{-\nu}$ and $(1 - u - v)^{-\nu}$ are given by

$$(1-u)^{-\nu} = \sum_{n=0}^{\infty} \frac{(\nu)_n u^n}{n!} \tag{1.4}$$

and

$$(1-u-v)^{-\nu} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\nu)_{n+k} u^n v^k}{n!\,k!} \tag{1.5}$$

The expansion (1.5) with $\nu = ½$ motivated M. A. Khan and M. P. Singh [4] to introduce two variable analogue of Legendre polynomials by taking $u = 2xs - s^2$ and $v = 2yt - t^2$ in (1.5). Thus they first attempted to define two variable analogues of polynomials by means of generating functions of the form G(u, v) where $u = 2xs - s^2$ and $v = 2yt - t^2$ before embarking on a particular example of it namely the two variable analogue of Legendre polynomial. In the present paper an attempt has been made to define and study a two variable analogue of Gegenbauer polynomials on the lines of [4]. We also recall here the following theorem due to M. A. Khan and M. P. Singh [4] :

Theorem 1. From

$$G\left(2xs-s^2, 2yt-t^2\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_{n,k}(x,y)s^n t^k$$

it follows that $\frac{\partial}{\partial x} g_{o,k}(x,y) = 0, k \geq 0, \frac{\partial}{\partial y} g_{n,0}(x,y) = 0, n \geq 0,$ and for $n, k \geq 1$,

$$x\frac{\partial}{\partial x}g_{n,k}(x,y)-n\,g_{n,k}(x,y)=\frac{\partial}{\partial x}g_{n-1,k}(x,y) \tag{1.6}$$

and

$$y\frac{\partial}{\partial y}g_{n,k}(x,y)-k\,g_{n,k}(x,y)=\frac{\partial}{\partial y}g_{n,k-1}(x,y) \tag{1.7}$$

Adding (1.6) and (1.7), we obtain

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)g_{n,k}(x,y)-(n+k)g_{n,k}(x,y)=\frac{\partial}{\partial x}g_{n-1,k}(x,y)+\frac{\partial}{\partial y}g_{n,k-1}(x,y) \tag{1.8}$$

The differential recurrence relations (1.6), (1,7) and (1.8) are common to all sets $g_{n,k}$ (x, y) possessing generating function of the form

$$G\left(2xs-s^2,2yt-t^2\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}g_{n,k}(x,y)s^n\,t^k \tag{1.9}$$

In this paper we shall consider the polynomial $g_{n,k}$ (x, y) for the choice $G(u, v) = (1 - u - v)^{-\nu}$.

2. The Gegenbauer Polynomials of Two Variables

We define the Gegenbauer polynomials of two variables denoted by $C^{\nu}_{n,k}(x,y)$ by the double generating relation

$$\left(1-2xs+s^2-2yt+t^2\right)^{-\nu}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C^{\nu}_{n,k}(x,y) \tag{2.1}$$

in which $(1 - 2xs + s^2 - 2yt + t^2)^{-\nu}$ denotes the particular branch which $\to 1$ as $s \to 0$ and $t \to 0$. We sha ll first show that $C^{\nu}_{n,k}(x,y)$ is a polynomial of degree precisely n in x and k in y.

Since $\left(1-u-v\right)^{-\alpha}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_{n+k}\,u^n\,v^k}{n!\,k!}$, we may write

$$\left(1-2xs+s^2-2yt+t^2\right)^{-\nu}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\nu)_{n+k}\left(2xs-s^2\right)^n\left(2yt-t^2\right)^k}{n!\,k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{k}\frac{(\nu)_{n+k}(2xs)^n(2yt)^k(-n)_r(-k)_j}{n!\,k!\,r!\,j!}\left(\frac{s}{2x}\right)^r\left(\frac{t}{2y}\right)^j$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \sum_{j=0}^{k} \frac{(\nu)_{n+k} (2x)^{n-r} (2y)^{k-j} (-1)^{r+j} s^{n+r} t^{k+j}}{r!\ j!\ (n-r)!\ (k-j)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(\nu)_{n+k-r-j} (2x)^{n-2r} (2y)^{k-2j} (-1)^{r+j} s^{n} t^{k}}{r!\ j!\ (n-2r)!\ (k-2j)!}$$

We thus obtain

$$C_{n,k}^{\nu}\ (x, y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} C_{n,k}^{\nu}\ (x, y)$$

$$= \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^{r+j}\ (\nu)_{n+k-r-j}\ (2x)^{n-2r}\ 2(2y)^{k-2j}}{r\ !\ j\ !\ (n-\ 2r)\ !\ (k-2j)\ !} \times$$

$$\frac{(-1)^{r+j}\ (\nu)_{n+k-r-j}\ (2x)^{n-2r}\ 2(2y)^{k-2j}}{r\ !\ j\ !\ (n-\ 2r)\ !\ (k-2j)\ !} \tag{2.2}$$

from which it follows that $C_{n,k}^{\nu}\ (x, y)$ is a polynomial in two variables x and y of degree precisely n in x and k in y. Thus $C_{n,k}^{\nu}\ (x, y)$ is a polynomial in two variables x and y of degree n + k. Equation (2.2) also yields

$$C_{n,k}^{\nu}\ (x, y) = \frac{2^{n+k} (\nu)_{n+k}\ x^{n}\ y^{k}}{n!\ k!} + \pi \tag{2.3}$$

where π is a polynomial in two variables x and y of degree n + k – 2.

If in (2.1), we replace x by – x and s by – s, the left member does not change. Hence

$$C_{n,k}^{\nu}\ (-x, y) = (-1)^{n}\ C_{n,k}^{\nu}\ (x, y) \tag{2.4}$$

Similarly by replacing y by – y and t by – t in (2.1), we obtain

$$C_{n,k}^{\nu}\ (x, -y) = (-1)^{K}\ C_{n,k}^{\nu}\ (x, y) \tag{2.5}$$

so that $C_{n,k}^{\nu}(x,y)$ is an odd function of x for n odd, an even function of x for n even. Similarly $C_{n,k}^{\nu}(x,y)$ is an odd function of y for k odd, an even function of y for k even.

Similarly replacing x by – x, y by – y, s by – s and t by – t in (2.1), we obtain

$$C_{n,k}^{\nu}(-x,-y)=(-1)^{n+k}C_{n,k}^{\nu}(x,y) \tag{2.6}$$

Putting t = 0 in (2.1), we get

$$C_{n,0}^{\nu}(x,y)=C_{n}^{\nu}(x) \tag{2.7}$$

where $C_{n}^{\nu}(x)$ is the well known Gegenbauer polynomial. Similarly by putting s = 0 in (2.1), we get

$$C_{0,k}^{\nu}(x,y)=C_{k}^{\nu}(y) \tag{2.8}$$

From (2.1) with x = 0 and y = 0, we get

$$\left(1+s^2+t^2\right)^{-\nu}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{\nu}(0,0)s^n t^k \tag{2.9}$$

But
$$\left(1+s^2+t^2\right)^{-\nu}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k}(\nu)_{n+k}\,s^{2n}\,t^{2k}}{n!\,k!}$$

Hence

$$\left.\begin{aligned}&C_{2n+1,2k}^{\nu}(0,0)=0,C_{2n,2k+1}^{\nu}(0,0)=0,C_{2n+1,2k+1}^{\nu}(0,0)=0,\\&C_{2n,2k}^{\nu}(0,0)=\frac{(-1)^{n+k}(\nu)_{n+k}}{n!\,k!}\end{aligned}\right\} \tag{2.10}$$

Equation (2.2) yields

$$\frac{\partial}{\partial x}C_{n,k}^{\nu}(x,y)=\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^{r+j}(\nu)_{n+k-r-j}\,2(2x)^{n-1-2r}(2y)^{k-2j}}{r!\,j!(n-1-2r)!(k-2j)!} \tag{2.11}$$

$$\frac{\partial}{\partial y}C_{n,k}^{\nu}(x,y)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\frac{(-1)^{r+j}(\nu)_{n+k-r-j}(2x)^{n-2r}\,2(2y)^{k-1-2j}}{r!\,j!(n-2r)!(k-1-2j)!} \tag{2.12}$$

$$\left.\begin{aligned}
\left[\frac{\partial}{\partial x} C^{\nu}_{2n+1,2k}(x,y)\right]_{x=0,y=0} &= \frac{(-1)^{n+k}\, 2(\nu)_{n+k+1}}{n!\,k!},\\
\left[\frac{\partial}{\partial x} C^{\nu}_{2n,2k}(x,y)\right]_{x=0,y=0} &= 0,\\
\left[\frac{\partial}{\partial x} C^{\nu}_{2n,2k+1}(x,y)\right]_{x=0,y=0} &= 0,\\
\left[\frac{\partial}{\partial x} C^{\nu}_{2n+1,2k+1}(x,y)\right]_{x=0,y=0} &= 0,
\end{aligned}\right\} \tag{2.13}$$

Similarly,

$$\left.\begin{aligned}
\left[\frac{\partial}{\partial y} C^{\nu}_{2n,2k}(x,y)\right]_{x=0,y=0} &= 0,\\
\left[\frac{\partial}{\partial y} C^{\nu}_{2n+1,2k}(x,y)\right]_{x=0,y=0} &= 0,\\
\left[\frac{\partial}{\partial y} C^{\nu}_{2n+1,2k+1}(x,y)\right]_{x=0,y=0} &= 0,\\
\text{and}\qquad & \\
\left[\frac{\partial}{\partial y} C^{\nu}_{2n,2k+1}(x,y)\right]_{x=0,y=0} &= \frac{(-1)^{n+k}\, 2(\nu)_{n+k+1}}{n!\,k!}
\end{aligned}\right\} \tag{2.14}$$

3. Differential Recurrence Relations

From Theorem 1, it is evident that the generating relation

$$\left(1-2xs+s^2-2yt+t^2\right)^{-\nu} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{n,k}(x,y)\, s^n t^k \tag{3.1}$$

implies the differential recurrence relations

$$x\frac{\partial}{\partial x}C^{\nu}_{n,k}(x,y)-nC^{\nu}_{n,k}(x,y)=\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(x,y), \tag{3.2}$$

$$y\frac{\partial}{\partial y}C^{\nu}_{n,k}(x,y)-kC^{\nu}_{n,k}(x,y)=\frac{\partial}{\partial y}C^{\nu}_{n,k-1}(x,y), \tag{3.3}$$

and

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C^{\nu}_{n,k}(x,y)-(n+k)C^{\nu}_{n,k}(x,y)=\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(x,y)+\frac{\partial}{\partial y}C^{\nu}_{n,k-1}(x,y), \tag{3.4}$$

From (3.1) it follows by the usual method (differentiation) that

$$2\nu\left(1-2xs+s^2-2yt+t^2\right)^{-\nu-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C^{\nu}_{n,k}(x,y)s^{n-1}t^{k} \tag{3.5}$$

$$2\nu\left(1-2xs+s^2-2yt+t^2\right)^{-\nu-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C^{\nu}_{n,k}(x,y)s^{n}t^{k-1} \tag{3.6}$$

$$2\nu(x-s)\left(1-2xs+s^2-2yt+t^2\right)^{-\nu-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}nC^{\nu}_{n,k}(x,y)s^{n-1}t^{k} \tag{3.7}$$

$$2\nu(y-t)\left(1-2xs+s^2-2yt+t^2\right)^{-\nu-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}kC^{\nu}_{n,k}(x,y)s^{n}t^{k-1} \tag{3.8}$$

Since $1-s^2-t^2-2s(x-s)-2t(y-t)=1-2sx+s^2-2yt+t^2$, we may multiply the left member of (3.5) by $1-s^2$, the left member of (3.6) by $-t^2$, the left member of (3.7) by $-2s$, the left member of (3.8) by $-2t$ and add and obtain the left member of (3.1). In this way we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C^{\nu}_{n,k}(x,y)\,s^{n-1}t^{k}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C^{\nu}_{n,k}(x,y)\,s^{n+1}t^{k}$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C^{\nu}_{n,k}(x,y)\,s^{n}t^{k+1}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}2nC^{\nu}_{n,k}(x,y)\,s^{n}t^{k}$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} 2kC_{n,k}^{\nu}(\mathrm{x},\mathrm{y})\, s^n t^k = 2\nu\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{\nu}(\mathrm{x},\mathrm{y})\, s^n t^k$$

or

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C_{n,k}^{\nu}(\mathrm{x},\mathrm{y})\, s^{n-1}t^k - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C_{n,k}^{\nu}(\mathrm{x},\mathrm{y})\, s^{n+1}t^k$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C_{n,k}^{\nu}(\mathrm{x},\mathrm{y})\, s^n t^{k+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2n+2k+2\nu)C_{n,k}^{\nu}(\mathrm{x},\mathrm{y})s^n t^k$$

We thus obtain another differential recurrence relation

$$(2n+2k+2\nu)C_{n,k}^{\nu}(\mathrm{x},\mathrm{y}) =$$

$$\frac{\partial}{\partial x}C_{n+1,k}^{\nu}(\mathrm{x},\mathrm{y}) - \frac{\partial}{\partial x}C_{n-1,k}^{\nu}(\mathrm{x},\mathrm{y}) - \frac{\partial}{\partial y}C_{n,k-1}^{\nu}(\mathrm{x},\mathrm{y}) \tag{3.9}$$

Similarly, we can get

$$(2n+2k+2\nu)C_{n,k}^{\nu}(\mathrm{x},\mathrm{y}) =$$

$$\frac{\partial}{\partial y}C_{n,k+1}^{\nu}(\mathrm{x},\mathrm{y}) - \frac{\partial}{\partial x}C_{n-1,k}^{\nu}(\mathrm{x},\mathrm{y}) - \frac{\partial}{\partial y}C_{n,k-1}^{\nu}(\mathrm{x},\mathrm{y}) \tag{3.10}$$

Adding (3.9) successively to (3.2), (3.3) and (3.4), we get

$$x\frac{\partial}{\partial x}C_{n,k}^{\nu}(\mathrm{x},\mathrm{y}) =$$

$$\frac{\partial}{\partial x}C_{n+1,k}^{\nu}(\mathrm{x},\mathrm{y}) - \frac{\partial}{\partial y}C_{n,k-1}^{\nu}(\mathrm{x},\mathrm{y}) - (n+2k+2\nu)C_{n,k}^{\nu}(\mathrm{x},\mathrm{y}) \tag{3.11}$$

$$y\frac{\partial}{\partial y}C_{n,k}^{\nu}(\mathrm{x},\mathrm{y}) =$$

$$\frac{\partial}{\partial x}C_{n+1,k}^{\nu}(\mathrm{x},\mathrm{y}) - \frac{\partial}{\partial x}C_{n-1,k}^{\nu}(\mathrm{x},\mathrm{y}) - (2n+k+2\nu)C_{n,k}^{\nu}(\mathrm{x},\mathrm{y}) \tag{3.12}$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C^{\nu}_{n,k}(x,y)=\frac{\partial}{\partial x}C^{\nu}_{n+1,k}(x,y)-(n+k+2\nu)C^{\nu}_{n,k}(x,y) \tag{3.13}$$

Adding (3.10) successively to (3.2), (3.3) and (3.4), we obtain

$$x\frac{\partial}{\partial x}C^{\nu}_{n,k}(x,y) =$$

$$\frac{\partial}{\partial y}C^{\nu}_{n,k+1}(x,y)-\frac{\partial}{\partial y}C^{\nu}_{n,k-1}(x,y)-(n+2k+2\nu)C^{\nu}_{n,k}(x,y) \tag{3.14}$$

$$y\frac{\partial}{\partial y}C^{\nu}_{n,k}(x,y) =$$

$$\frac{\partial}{\partial y}C^{\nu}_{n,k+1}(x,y)-\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(x,y)-(2n+k+2\nu)C^{\nu}_{n,k}(x,y) \tag{3.15}$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C^{\nu}_{n,k}(x,y)=\frac{\partial}{\partial y}C^{\nu}_{n,k+1}(x,y)-(n+k+2\nu)C^{\nu}_{n,k}(x,y) \tag{3.16}$$

Shifting the index from n to n – 1 in (3.11) and using (3.2), we get

$$\left(x^2-1\right)\frac{\partial}{\partial x}C^{\nu}_{n,k}(x,y)=$$

$$nxC^{\nu}_{n,k}(x,y)-\frac{\partial}{\partial y}C^{\nu}_{n-1,k-1}(x,y)-(n+2k+2\nu-1)C^{\nu}_{n-1,k}(x,y) \tag{3.17}$$

Similarly shifting the index from k to k – 1 in (3.15) and using (3.3), we get

$$\left(y^2-1\right)\frac{\partial}{\partial y}C^{\nu}_{n,k}(x,y)=$$

$$nxC^{\nu}_{n,k}(x,y)-\frac{\partial}{\partial y}C^{\nu}_{n-1,k-1}(x,y)-(n+2k+2\nu-1)C^{\nu}_{n-1,k}(x,y) \tag{3.18}$$

Adding (3.17) and (3.18) we obtain

$$\left\{\left(x^2-1\right)\frac{\partial}{\partial x}+\left(y^2-1\right)\frac{\partial}{\partial y}\right\}C^{\nu}_{n,k}(x,y)=$$

$$(nx+ky)C-\left\{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right\}C^{\nu}_{n-1,k-1}(\mathrm{x,y})^{\nu}_{n,k}(\mathrm{x,y})-(n+2k+2\nu-1)C^{\nu}_{n-1,k}(\mathrm{x,y})$$

$$-(2n+k+2\nu-1)C^{\nu}_{n,k-1}(\mathrm{x,y}) \tag{3.19}$$

4. Partial differential Equation of $\mathbf{C^{\nu}_{n,k}(x,y)}$

From (3.2) and (3.3), we have

$$\left.\begin{aligned}
&\frac{\partial}{\partial \mathrm{x}}\mathrm{C}^{\nu}_{\mathrm{n-1,k}}(\mathrm{x,y})=\mathrm{x}\frac{\partial}{\partial \mathrm{x}}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})-\mathrm{n}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})\\
&\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{C}^{\nu}_{\mathrm{n-1,k}}(\mathrm{x,y})=\mathrm{x}\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})+(1-\mathrm{n})\frac{\partial}{\partial \mathrm{x}}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})
\end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial \mathrm{y}}\mathrm{C}^{\nu}_{\mathrm{n,k-1}}(\mathrm{x,y})=\mathrm{y}\frac{\partial}{\partial \mathrm{y}}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})-\mathrm{k}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})\\
&\frac{\partial^2}{\partial \mathrm{y}^2}\mathrm{C}^{\nu}_{\mathrm{n,k-1}}(\mathrm{x,y})=\mathrm{y}\frac{\partial^2}{\partial \mathrm{y}^2}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})+(1-\mathrm{k})\frac{\partial}{\partial \mathrm{y}}\mathrm{C}^{\nu}_{\mathrm{n,k}}(\mathrm{x,y})
\end{aligned}\right\} \tag{4.2}$$

Shifting the index from n to n –1 in (3.11) and from k to k – 1 in (3.15), we get

$$x\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(\mathrm{x,y})=$$

$$\frac{\partial}{\partial x}C^{\nu}_{n,k}(\mathrm{x,y})-(n+2k+2\nu-1)C^{\nu}_{n-1,k}(\mathrm{x,y})-\frac{\partial}{\partial y}C^{\nu}_{n-1,k-1}(\mathrm{x,y}) \tag{4.3}$$

$$y\frac{\partial}{\partial y}C^{\nu}_{n,k-1}(\mathrm{x,y})=$$

$$\frac{\partial}{\partial y}C^{\nu}_{n,k}(\mathrm{x,y})-(2n+k+2\nu-1)C^{\nu}_{n,k-1}(\mathrm{x,y})-\frac{\partial}{\partial x}C^{\nu}_{n-1,k-1}(\mathrm{x,y}) \tag{4.4}$$

Differentiation (4.3) partially with respect to x and (4.4) with respect to y, we get

$$x\,\frac{\partial^2}{\partial x^2}C^{\nu}_{n-1,k}(x,y)=$$

$$\frac{\partial^2}{\partial x^2}C^{\nu}_{n,k}(x,y)-(n+2k+2\nu)\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(x,y)-\frac{\partial^2}{\partial x\,\partial y}C^{\nu}_{n-1,k-1}(x,y) \qquad (4.5)$$

$$y\,\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k-1}(x,y)=$$

$$\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k}(x,y)-(2n+k+2\nu)\frac{\partial}{\partial y}C^{\nu}_{n,k-1}(x,y)-\frac{\partial^2}{\partial x\,\partial y}C^{\nu}_{n-1,k-1}(x,y) \qquad (4.6)$$

Using (4.1) in (4.5) and (4.2) in (4.6), we obtain

$$(1-x^2)\frac{\partial^2}{\partial x^2}C^{\nu}_{n,k}(x,y)-(2k+2\nu+1)x\,\frac{\partial}{\partial x}C^{\nu}_{n,k}(x,y)+n(n+2k+2\nu)C^{\nu}_{n,k}(x,y)$$

$$-\frac{\partial^2}{\partial x\,\partial y}C^{\nu}_{n-1,k-1}(x,y)=0 \qquad (4.7)$$

$$(1-y^2)\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k}(x,y)-(2n+2\nu+1)y\,\frac{\partial}{\partial y}C^{\nu}_{n,k}(x,y)+k(2n+k+2\nu)C^{\nu}_{n,k}(x,y)$$

$$-\frac{\partial^2}{\partial x\,\partial y}C^{\nu}_{n-1,k-1}(x,y)=0 \qquad (4.8)$$

Subtracting (4.8) from (4.7), we obtain

$$\left\{(1-x^2)\frac{\partial^2}{\partial x^2}-(1-y^2)\frac{\partial^2}{\partial y^2}\right\}C^{\nu}_{n,k}(x,y)-\left\{(2k+2\nu+1)x\frac{\partial}{\partial x}-(2n+2\nu+1)y\frac{\partial}{\partial y}\right\}C^{\nu}_{n,k}(x,y)$$

$$+(n-k)(n+k+2\nu)\,C^{\nu}_{n,k}(x,y)=0 \qquad (4.9)$$

Here (4.9) is the partial differential equation satisfied by $C^{\nu}_{n,k}(x,y)$.

5. Additional double Generating Functions

The generating function $(1-2xs+s^2-2yt+t^2)^{-\nu}$ used to define a polynomial $C^{\nu}_{n,k}(x,y)$ in two variables x and y analogues to Gegenbauer polynomials $C^{\nu}_{n}(x)$ in a single variable x can be expanded in powers of s and t in new ways, thus yielding additional results. For instance

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{\nu}(x,y)s^n t^k = \left(1-2xs+s^2-2yt+t^2\right)^{-\nu}$$

$$= \left[(1-xs-yt)^2 - s^2\left(x^2-1\right) - t^2\left(y^2-1\right) - 2xyst\right]^{-\nu}$$

$$= (1-xs-yt)^{-2\nu}\left[1-\frac{s^2\left(x^2-1\right)}{(1-xs-yt)^2}-\frac{t^2\left(y^2-1\right)}{(1-xs-yt)^2}-\frac{2xyst}{(1-xs-yt)^2}\right]^{-\nu}$$

$$=\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\nu)_{j+p+r}\, s^{2j}\left(x^2-1\right)^j t^{2p}\left(y^2-1\right)^p (2xyst)^r}{j!\,p!\,r!\,(1-xs-yt)^{2j+2p+2r+2\nu}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(\nu)_{j+p+r}\;(2\nu+2j+2p+2r)_{n+k}$$

$$\times\frac{s^{n+2j+r}\left(x^2-1\right)^j\; t^{k+2p+r}\left(y^2-1\right)^p\; 2^r x^{n+r}y^{k+r}}{j!\;p!\;r!\;n!\;k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(\nu)_{j+p+r}\;(2\nu)_{n+k+2j+2p+2r}$$

$$\times\frac{s^{n+2j+r}\left(x^2-1\right)^j\; t^{k+2p+r}\left(y^2-1\right)^p\; 2^r x^{n+r}y^{k+r}}{j!\,p!\,r!\,n!\,k!\;(2\nu)_{2j+2p+2r}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{(2\nu)_{n+k}\left(x^2-1\right)^j\left(y^2-1\right)^p s^n t^k x^{n-2j}y^{k-2p}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\,p!\,r!\,(n-r-2j)!\,(k-r-2p)!}\;.$$

Equating the coefficients of $s^n t^k$, we obtain

$$C_{n,k}^{\nu}(x,y)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{(2\nu)_{n+k}!\left(x^2-1\right)^j\left(y^2-1\right)^p x^{n-2j}y^{k-2p}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\,p!\,r!\,(n-r-2j)!\,(k-r-2p)!} \qquad (5.1)$$

Let us now employ (5.1) to discover new double generating function for $C^{\nu}_{n,k}(x,y)$. Consider, for arbitrary c, the double sum

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}\,C^{\nu}_{n,k}(x,y)s^n t^k}{(2\nu)_{n+k}!}$$

$$=\sum_{n,k=0}^{\infty}\frac{(c)_{n+k}\,s^n t^k}{(2\nu)_{n+k}}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{(2\nu)_{n+k}\,(x^2-1)^j(y^2-1)^p\,x^{n-2j}y^{k-2p}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\,p!\,r!\,(n-r-2j)!\,(k-r-2p)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(c)_{n+k+2j+2p+2r}\,(x^2-1)^j(y^2-1)^p\,x^{n+r}y^{k+r}s^{n+2j+r}\,t^{k+2p+r}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\;p!\;r!\;n!\;k!}$$

$$=\sum_{j,p,r=0}^{\infty}\frac{(c)_{2j+2p+2r}\,(x^2-1)^j(y^2-1)^p\,(xy)^r s^{2j+r}\,t^{2p+r}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\;p!\;r!}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c+2j+2p+2r)_{n+k}\,(xs)^n(yt)^k}{n!\,k!}$$

$$=\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(c)_{2j+2p+2r}(x^2-1)^j(y^2-1)^p\,(xy)^r s^{2j+r}\,t^{2p+r}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\;p!\;r!}(1-xt-yt)^{-c-2j-2p-2r}$$

$$=(1-xs-yt)^{-c}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\frac{c}{2}\right)_{j+p+r}\left(\frac{c}{2}+\frac{1}{2}\right)_{j+p+r}}{\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\;p!\;r!}\left\{\frac{s^2(x^2-1)}{(1-xs-yt)^2}\right\}^j$$

$$\times\left\{\frac{t^2(y^2-1)}{(1-xs-yt)^2}\right\}^p\left\{\frac{2xyst}{(1-xs-yt)^2}\right\}^r$$

$$=(1-xs-yt)^{-c}\,F^{(3)}\left[\begin{matrix}\frac{c}{2},\frac{c}{2}+\frac{1}{2}::-;-;-:-;-;-; \\ \\ \frac{2\nu+1}{2}\;::-;-;-:-;-;-;\end{matrix}\;\frac{s^2(x^2-1)}{(1-xs-yt)^2},\frac{t^2(y^2-1)}{(1-xs-yt)^2},\frac{2xyst}{(1-xs-yt)^2}\right]$$

where $F^{(3)}[x,y,z]$ is a triple hypergeometric series [cf. Srivastava [2], p.428].
we have thus discovered the family of double generating functions

$$(1-xs-yt)^{-c}\,F^{(3)}\left[\begin{matrix}\frac{c}{2},\frac{c}{2}+\frac{1}{2}::-;-;-:-;-;-; \\ \frac{2\nu+1}{2}::-;-;-:-;-;-;\end{matrix}\ \frac{s^2(x^2-1)}{(1-xs-yt)^2},\frac{t^2(y^2-1)}{(1-xs-yt)^2},\frac{2xyst}{(1-xs-yt)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}\,C_{n,k}^{\nu}(x,y)s^n t^k}{(2\nu)_{n+k}} \tag{5.2}$$

In which c may be any complex number.

Let us now return to (5.1) and consider the double sum

$$\sum_{n,k=0}^{\infty}\frac{C_{n,k}^{\nu}(x,y)s^n t^k}{(2\nu)_{n+k}}\sum_{n,k=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{r=0}^{\min(n,k)}\frac{(x^2-1)^j(y^2-1)^p\,x^{n-2j}y^{k-2p}\,s^n t^k}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r}j!p!r!(n-2j-r)!(k-2p-r)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(x^2-1)^j(y^2-1)^p\,x^{n+r}y^{k+r}s^{n+2j+r}\,t^{k+2p+r}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r}j!\,p!\,r!\,n!\,k!}$$

$$=\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(x^2-1)^j(y^2-1)^p\,(xyst)^r s^{2j}\,t^{2p}}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r}j!\,p!\,r!}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(xs)^n(yt)^k}{n!k!}$$

$$=e^{xs+yt}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{j!\,p!\,r!\left(\frac{2\nu+1}{2}\right)_{j+p+r}}\left\{\frac{s^2(x^2-1)}{4}\right\}^j,\left\{\frac{t^2(y^2-1)}{4}\right\}^p\left\{\frac{xyst}{2}\right\}^r$$

$$=e^{xs+yt}\,F^{(3)}\left[\begin{matrix}-::-;-;-:-;-;-; \\ \frac{2\nu+1}{2}::-;-;-:-;-;-;\end{matrix}\ \frac{s^2(x^2-1)}{4},\frac{t^2(y^2-1)}{4},\frac{xyst}{2}\right] \tag{5.3}$$

6. Triple Hypergeometric Forms of $C_{n,k}^{\nu}(x,y)$

We return once more to the original definition of $C_{n,k}^{\nu}(x,y)$:

$$\left(1-2xs+s^2-2yt+t^2\right)^{-\nu}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{\nu}(x,y)s^n t^k \tag{6.1}$$

This time we note that

$$\left(1-2xs+s^2-2yt+t^2\right)^{-\nu}=\left[(1-s-t)^2-2s(x-1)-2t(y-1)-2st\right]^{-\nu}$$

$$=(1-s-t)^{-2\nu}\left[1-\frac{2s(x-1)}{(1-s-t)^2}-\frac{2t(y-1)}{(1-s-t)^2}-\frac{2st}{(1-s-t)^2}\right]^{-\nu}$$

which permits us to write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{\nu}(x,y)s^n t^k=\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{2^{j+p+r}(\nu)_{j+p+r}(x-1)^j(y-1)^p s^{j+r}t^{p+r}}{j!\,p!\,r!\,(1-s-t)^{2j+2p+2r+2\nu}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{2^{j+p+r}(\nu)_{j+p+r}(x-1)^j(y-1)^p(2\nu)_{2j+2p+2r+n+k}s^{n+j+r}t^{k+p+r}}{j!\,p!\,r!\,n!\,k!\,(2\nu)_{2j+2p+2r}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\sum_{p=0}^{k}\sum_{r=0}^{\min(n,k)}\frac{2^{j+p+r}(\nu)_{j+p+r}(x-1)^j(y-1)^p(2\nu)_{n+k+j+p}s^n t^k}{j!\,p!\,r!\,(n-j-r)!\,(k-p-r)!\,(2\nu)_{2j+2p+2r}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\sum_{p=0}^{k}\sum_{r=0}^{\min(n,k)}\frac{(2\nu)_{n+k+j+p}(x-1)^j(y-1)^p s^n t^k}{2^{j+p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r}j!\,p!\,r\;(n-j-r)!\,(k-p-r)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\sum_{p=0}^{k}\sum_{r=0}^{\min(n,k)}\frac{(2\nu)_{n+k}\,(2\nu+n+k)_{j+p}(-n)_{j+r}(-k)_{p+r}(1-x)^j(1-y)^p s^n t^k}{2^{j+p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r}j!\,p!\,r!\;n!\;k!}$$

Therefore

$$C^{\nu}_{n,k}(x,y) = \frac{(2\nu)_{n+k}}{n!\,k!} F^{(3)}\left[\begin{matrix} -::-n;-k;2\nu+n+k:-;-;-; \\ \qquad\qquad\qquad\qquad \frac{1-x}{2}, \frac{1-y}{2}, \frac{1}{2} \\ \frac{2\nu+1}{2}::-;-;-:-;-;-; \end{matrix}\right] \tag{6.2}$$

Since $C^{\nu}_{n,k}$ (–x, –y) = $(-1)^{n+k}$ $C^{\nu}_{n,k}(x,y)$, it follows from (6.2) that also

$$C^{\nu}_{n,k}(x,y) = \frac{(-1)^{n+k}(2\nu)_{n+k}}{n!\,k!} F^{(3)}\left[\begin{matrix} -::-n;-k;2\nu+n+k:-;-;-; \\ \qquad\qquad\qquad\qquad \frac{1+x}{2}, \frac{1+y}{2}, \frac{1}{2} \\ \frac{2\nu+1}{2}::-;-;-:-;-;-; \end{matrix}\right] \tag{6.3}$$

Next, consider (2.2) again

$$C^{\nu}_{n,k}(x,y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^{r+j}(\nu)_{n+k-r-j}(2x)^{n-2r}(2y)^{k-2j}}{r!\,j!\,(n-2r)!(k-2j)!}$$

We may write it as

$$C^{\nu}_{n,k}(x,y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(-n)_{2r}(-k)_{2j}(\nu)_{n+k}(2x)^{n-2r}(2y)^{k-2j}(-1)^{r+s}}{r!\,j!\,n!\,k!\,(\nu-n-k)_{r+j}}$$

$$= \frac{2^{n+k}(\nu)_{n+k}x^n y^k}{n!\,k!} \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{\left(-\frac{n}{2}\right)_r \left(-\frac{n}{2}+\frac{1}{2}\right)_r \left(-\frac{k}{2}\right)_j \left(-\frac{k}{2}+\frac{1}{2}\right)_j (-1)^{r+s}}{r!\,j!\,(\nu-n-k)_{r+j}\,x^{2r}y^{2j}}$$

or in terms of Kampe de Feriet’s double hypergeometric function, we have

$$C^{\nu}_{n,k} = \frac{2^{n+k}(\nu)_{n+k}x^n y^k}{n!\,k!} F\left[\begin{matrix} -:-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};-\frac{k}{2},-\frac{k}{2}+\frac{1}{2}; \\ \qquad\qquad\qquad\qquad -\frac{1}{x^2}, -\frac{1}{y^2} \\ \nu-n-k:-;-; \end{matrix}\right] \tag{6.4}$$

7. A Special Property of $C^{\nu}_{n,k}(x, y)$

We now return to the original definition of $C^{\nu}_{n,k}(x, y)$ and for convenience use $\rho=(1 - 2xs + s^2 - 2yt + t^2)^{\nu}$. We know that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{n,k}(x, y)s^n t^k = \rho^{-1} \tag{7.1}$$

In (7.1), we replace x by $\frac{x-s}{\rho}$, by $\frac{y-t}{\rho}$, s by $\frac{u}{\rho}$ and t by $\frac{v}{\rho}$ to get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{n,k}\left(\frac{x-s}{\rho},\frac{y-t}{\rho}\right)\rho^{-n-k}u^n v^k = \left[1-\frac{2(x-s)u}{\rho^2}+\frac{u^2}{\rho^2}-\frac{2(y-t)v}{\rho^2}+\frac{v^2}{\rho^2}\right]^{-\nu}$$

$$= \rho^{2\nu}\left[\rho^2 - 2(x-s)u + u^2 - 2(y-t)v + v^2\right]^{-\nu}$$

We may now write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{n,k}\left(\frac{x-s}{\rho},\frac{y-t}{\rho}\right)\rho^{-n-k-2\nu}u^n v^k$$

$$= \left[1 - 2xs + s^2 - 2yt + t^2 - 2xu + 2us + u^2 - 2yv + 2vt + v^2\right]^{-\nu}$$

$$= \left[1 - 2x(s+u) + (s+u)^2 - 2y(t+v) + (t+v)^2\right]^{-\nu}$$

which by (7.1) yields

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{n,k}\left(\frac{x-s}{\rho},\frac{y-t}{\rho}\right)\rho^{-n-k-2\nu}u^n v^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{n,k}(x, y)(s+u)^n (t+v)^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{k} \frac{n!k!C^{\nu}_{n,k}(x, y)s^r u^{n-r}\, t^j v^{k-j}}{r!j!(n-r)!(k-j)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \frac{(n+r)!(k+j)!C^{\nu}_{n+r,k+j}(x, y)s^r\, t^j u^n v^k}{r!j!n!k!}$$

Equation the coefficients of $u^n v^k$ in the above, we find that

$$\rho^{-n-k-2\nu} C^{\nu}_{n,k}\left(\frac{x-s}{\rho}, \frac{y-t}{\rho}\right) = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+r)!(k+j)! C^{\nu}_{n+r,k+j}(x,y) s^r t^j}{r!j!n!k!} \tag{7.2}$$

in which $\rho = (1 - 2xs + s^2 - 2yt + t^2)^{\nu}$.

8. More Generating Functions

As an example of the use of equation (7.2), we shall apply (7.2) to the generating relation

$$e^{xs+yt} F^{(3)}\left[\begin{matrix} - :: -; -; -: -; -; -; \\ \frac{2\nu+1}{2} :: -; -; -: -; -; -; \end{matrix} \frac{s^2(x^2-1)}{4}, \frac{t^2(y^2-1)}{4}, \frac{xyst}{2}\right] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{C^{\nu}_{n,k}(x,y) s^n t^k}{(2\nu)_{n+k}} \tag{8.1}$$

In (8.1), we replace x by $\frac{x-s}{\rho}$, y by $\frac{y-t}{\rho}$, s by $\frac{-su}{\rho}$ and t by $\frac{-tv}{\rho}$ and multiply each member by $\rho^{-2\nu}$ where $\rho = (1 - 2xs + s^2 - 2yt + t^2)^{\nu}$, we obtain

$$\rho^{-2\nu} \exp\left(-\frac{su(x-s)+tv(y-t)}{\rho^2}\right) \times$$

$$F^{(3)}\left[\begin{matrix} - :: -; -; -: -; -; -; \\ \frac{2\nu+1}{2} :: -; -; -: -; -; -; \end{matrix} \frac{u^2 s^2(x^2-1+2yt-t^2)}{4\rho^4}, \frac{v^2 t^2(y^2-1+2xs-s^2)}{4\rho^4}, \frac{uvst(x-s)(y-t)}{2\rho^4}\right]$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k} \rho^{-n-k-2\nu} C^{\nu}_{n,k}\left(\frac{x-s}{\rho}, \frac{y-t}{\rho}\right) s^n u^n t^k v^k}{(2\nu)_{n+k}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{n+k} (n+r)!(k+j)! C^{\nu}_{n+r,k+j}(x,y) s^{n+r} u^n t^{k+j} v^k}{n!k!r!j!(2\nu)_{n+k}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \sum_{j=0}^{k} \frac{(-1)^{n+k-r-j} n! k! C_{n,k}^{\nu}(x,y) u^{n-r} v^{k-j} s^n t^k}{(n-r)!(k-j)! r! j! (2\nu)_{n+k-r-j}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \sum_{j=0}^{k} \frac{(-1)^{r+j} n! k! C_{n,k}^{\nu}(x,y) u^{r} v^{j} s^n t^k}{r! j! (n-r)!(k-j)! (2\nu)_{r+j}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \sum_{j=0}^{k} \frac{(-n)_r (-k)_j C_{n,k}^{\nu}(x,y) u^{r} v^{j} s^n t^k}{r! j! (2\nu)_{r+j}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Phi_2[-n,-k; 2\nu; u, v] C_{n,k}^{\nu}(x,y) s^n t^k$$

where Φ_2 is one of the seven confluent forms of the four Appell series defined by Humbert (see [3], pp. 45). This gives a bilinear double generating function.

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