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Some Geometrical Results about the Convex Deficiency of a Compact Set

Víctor H. González

Universidad Nacional de General Sarmiento
Instituto del Desarrollo Humano
Los Polvorines, Bs. As. Argentina
vgonzale@ungs.edu.ar

Mabel A. Rodríguez

Universidad Nacional de General Sarmiento
Instituto del Desarrollo Humano
Los Polvorines, Bs. As. Argentina
mrodri@ungs.edu.ar

Abstract

The convex deficiency has been recently used to characterize the convex kernel of a set. This kind of result, central for Visibility Theory, suggests to study the convex deficiency in terms of some Visibility elements. The aim of this work is to provide some geometrical and topological results about the convex deficiency of a compact set in the n-dimensional euclidian space.

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1 Introduction

The convex deficiency of a set was studied by different authors (see for example [1], [2], [3], [4]). In these articles it is presented a geometrical study of the convex deficiency in relation with skeletal points and skeletons. These three notions have been recently used in [5] to obtain a characterization of the convex kernel of a closed set by means of the stars of certain spherical points.

In this paper we include some topological results referred to the convex deficiency of a set in R^n, and some others in terms of Theory Visibility.

Unless otherwise stated, all the points and sets considered here are included in R^n the real n-dimensional euclidian space whose origin is denoted θ. The open segment joining a and b is denoted (a, b), and the substitution of one or both parenthesis by square brackets indicates the adjunction of the corresponding extremes. The interior, closure, boundary, and complement of a set K are denoted by: $intK$, clK, ∂K, K^C respectively. The *convex hull* of a set K is denoted $convK$, and the *convex deficiency* of a set K, denoted $D(K)$, is $D(K) = convK \backslash K$. The closed ray issuing from x and going through y is written $R(x \to y)$. A *convex body* is a closed convex set with $intK \neq \emptyset$. $B(x, \varepsilon)$ denotes the open ball with x as center and having radius $\varepsilon > 0$.

A few basic definitions from Visibility Theory are included here. A point x of a set K *sees via* K $y \in K$ if $[x, y] \subset K$. The *star* of a point $p \in K$ is the set $st(p, K) = \{x \in K/[x, p] \subset K\}$. The *convex kernel* of a set K is $\ker K = \{x \in K/st(x, K) = K\}$. A point p of a convex set K is called an *extreme point* of K if it does not belong to any open segment having its endpoints lying on K. A point $p \in K$ is a point of *local convexity* of K if there exists a neighborhood U of p such that $U \cap K$ is a convex set. In other case it is a *local nonconvexity* point. The set of all the points of local convexity and local nonconvexity are denoted lcK and $lncK$ respectively. A *regular point* p of a set K is a point such that there exists an hyperplane which contains the star of p in one of the closed half-spaces determined by it. The *inscribed cone* to a set A from the point a ($a \in A$) is $I(A, a) = \{a\} \cup \{t \in R(a \to x)/R(a \to x) \subset A\}$. The *infinitude cone* of A is $I(A) = \{\theta\} \cup \{t \in R(a \to x) - a/R(a \to x) \subset A$ for some $a \in A\}$.

2 Main Results

We begin this section with a list of some topological results referred to the convex deficiency of a set in R^n which can be easily proved by means of standard arguments.

Lemma 2.1 *Let $K \subset R^n$ be a nonconvex compact set and K_0 a connected component of $D(K)$. Let be $x \in \partial K_0$ then it holds that: $x \notin (convK)^C$; $x \in \partial K$ or $x \in \partial convK$; $x \notin intK$. Besides, if $x \notin \partial K$ then $x \in K^C$.*

Lemma 2.2 *Let $K \subset R^n$ be a bounded convex body, then K^C is a path-connected set.*

Lemma 2.3 *Let $K \subset R^n$ be a nonconvex closed set and K_0 connected component of $D(K)$ such that $\partial K_0 \subset \partial K$. Then K_0 is an open set.*

Lemma 2.4 *Let $K \subset R^n$ be a nonconvex closed set. Any point of local nonconvexity of K belongs to the boundary of the convex deficiency of the set.*

Theorem 2.5 *Let $K \subset R^n$ be a nonconvex compact set and K_0 a connected component of $D(K)$. If there exists K_1 a bounded connected component of K^C such that K_1 contains K_0, then it holds that: $\partial K_0 \subset \partial K$ and any closed ray issuing from every $x \in K_0$ verifies that it cannot be wholly included in K^C.*

Proof. It is immediate that $\partial K_0 \subset \partial K$. Let $x \in K_0$, as K_0 is an open set, there exists $U_x = B(x, \varepsilon) \subset K_0$. Any closed ray issuing from x can be described as $R(x \to q)$ where $q \in \partial U_x$. Suppose that $R(x \to q) \cap K = \emptyset$. The fact that $R(x \to q)$ is wholly included in K^C means that the points of K_0 and $(convK)^C$ would belong to the same path-connected component. Hence K_0 will be included in an unbounded connected component of K^C, a contradiction.

The following theorem explores one relation between the convex deficiency of a compact set and the useful tool of cones developed in its complement. It is clear that the infinitude cone of the complement of a compact set is the whole space. What we prove here is that some of the half-lines that compose this cone have origin at a point of the convex deficiency of the compact set.

Theorem 2.6 *Let $K \subset R^n$ be a nonconvex compact set and K_0 a connected component of $D(K)$. Let S be the connected component of K^C such that S contains K_0. We suppose that $\partial K_0 \not\subset \partial K$, then there exists $x_0 \in K_0$ such that $I(S, x_0) \neq \{x_0\}$.*

Proof. The fact that $\partial K_0 \not\subset \partial K$ implies that we can take $f : [0;1] \to K^C$ continuous such that $f(0) = x$ and $f(1) = q$ for any $x \in K_0$ and $q \in (convK)^C$. Let $x_0 \in Imf \cap \partial convK$ and H a support hyperplane of $convK$ that passes through x_0. Denote H^+ and H^- the closed half-spaces determined by H where H^+ contains $convK$. Let $t \in H^-$, it follows easily that $R(x_0 \to t) \subset K^C$. Then, $R(x_0 \to t) \subset I(S, x_0)$.

Next result follows easily by Milman‘s theorem (see [6] pg. 49).

Theorem 2.7 *Let $K \subset R^n$ be a nonconvex compact set and we denote P the set of extreme points of $convK$. Then $\partial convK \cap (\partial K)^C \cap P = \emptyset$.*

The convex deficiency of a nonconvex compact set presents a “flat part”, what is precisely stated as follows.

Theorem 2.8 *Let $K \subset R^n$ be a nonconvex compact set and K_0 a connected component of $D(K)$. If ∂K_0 is not included in ∂K, then there exist $y, z \in \partial convK$ such that $[y, z] \subset \partial convK$.*

Proof. Let $x \in K_0$ and $q \in (convK)^C$. The argument used in the proof of theorem 2.6 lets us assert that there exists $f : [0;1] \to K^C$ continuous, such that $f(0) = x$ and $f(1) = q$. Let us take $x_0 \in Imf \cap \partial convK$, then $x_0 \in convK$. We consider P the set of extreme points of $convK$. Recall that the previous result implies that $x_0 \notin P$ since $x_0 \in \partial convK \cap (\partial K)^C$. We know by Minkowski‘s theorem that $convK = convP$, then being x_0 in $\partial convK$ it results that $x_0 \in \partial convP$. The fact that $x_0 \notin P$ but it does belong to its boundary implies that it lies in a certain segment with extremes in $convP$, then $x_0 \in [y, z]$ where $y, z \in \partial convP$. It follows easily that $[y, z] \subset \partial convK$.

This statement cannot be improved following this idea using simplices instead of segments as the following example shows. The set proposed is $A = A_1 \cup A_2 \cup A_3$ where $A_1 = \{(x, y, z) \in R^3/x^2 + y^2 \leq 4 \text{ and } 0 \leq z \leq 1\}$, $A_2 = \{(x, y, z) \in R^3/x^2 + y^2 \leq 4 \text{ and } 2 \leq z \leq 3\}$ and $A_3 = \{(x, y, z) \in R^3/x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 3\}$. Notice that A is closed, connected and $cl\ intA = A$ and its convex deficiency is $D(A) = \{(x, y, z) \in R^3/1 < x^2 + y^2 \leq 4 \text{ and } 1 < z < 2\}$ which does not contain any simplex -except segments- in its boundary.

This last result is connected with the well known problem "given two convex sets A and B such that $A \subset B$, under which conditions there exists some set S such that $A = kerS$ and $B = convS$?". Our theorem imposes a condition for B, necessarily it has to present a flat part.

The next result characterizes the star of certain points of local nonconvexity that lie in the closure of the convex deficiency of the set. It is related with regular points because our theorem describes completely the star, instead of stating in which half-space it lies.

Theorem 2.9 *Let $K \subset R^n$ be a nonconvex closed set whose convex deficiency contains only one connected component K_0. Suppose that there exists $p \in \partial convclK_0 \cap \partial K \cap intconvK$. Let H be a support hyperplane of $convclK_0$ through p. Denote H^+ the closed half-space determined by H such that H^+ does not contain K_0. If $p \in lcK$, then $H^+ \cap K = st(p, K)$.*

Proof. Let $p \in \partial convclK_0 \cap \partial K \cap intconvK$. We take H a support hyperplane of $convclK_0$ through p. Let H^+ be the closed half-space determined by H such that $H^+ \not\supset K_0$ and H^- its closed complement. As p belongs to $intconvK$, using a well known result, (theorem 3.11, pg. 40, [7]) $intH^+ \cap K \neq \emptyset$ and $intH^- \cap K \neq \emptyset$.

Let us suppose that $H^+ \cap K$ is a convex set. Then it holds that $st(p, H^+ \cap K) = H^+ \cap K$. It is immediate that $st(p, H^+ \cap K) \subset st(p, K)$. To prove the other inclusion, suppose that there exists $t \in st(p, K)$ such that $[p, t] \not\subset (H^+ \cap K)$. As t belongs to $st(p, K)$, it holds that $[p, t] \subset K$, but since this segment is not included in $H^+ \cap K$, it results that $[p, t] \not\subset H^+$. We state that $t \in H^-$. As $p \in lcK$ we pick U_p a closed neighborhood of p such that $U_p \cap K$ is convex. Without loss of generality we assume that $t \in \partial U_p$. Consider the set $D = conv(\{t\} \cup (U_p \cap K \cap H^+))$ which is included in K. Then $p \in intD \subseteq intK$. This produces a contradiction.

In short $st(p, H^+ \cap K) = H^+ \cap K$ and $st(p, H^+ \cap K) = st(p, K)$, then, $H^+ \cap K = st(p, K)$. It remains to prove that $H^+ \cap K$ is a convex set. It is immediate that $intH^+$ is a convex set and so is $intH^+ \cap convK$. This set is nonempty since $intH^+ \cap K \neq \emptyset$. The fact that K_0 is the only connected component of $D(K)$ allows us to affirm that $convK = K \cup K_0$ and as K_0 is included in H^- it holds that $intH^+ \cap K_0 = \emptyset$. Then $intH^+ \cap convK = intH^+ \cap K$. This last statement implies that $intH^+ \cap K$ is a convex set and also is its closure and the assumption that $H^+ \cap K$ is convex set holds.

Finally we make two remarks about the requirements for the point p trying to show that the condition stated is not neither too general that every point satisfies it, nor too restrictive, so that it is not satisfied by any point. Consider the set $A = \{(x, y) \in R^2/(x-1)^2 + y^2 \leq 1$ and $y \leq 0\} \cup \{(x, y) \in R^2/y = 0$ and $2 \leq x \leq 3\} \cup \{(x, y) \in R^2/x = 0$ and $-3 \leq y \leq 0\}$. In this case it does not exist such a point p even though A is a compact set. On the other hand, even considering a set that is not connected, there exist points p in the conditions of the statement. Take for example the set $C = [0, 2] \times [0, 2] \cup \{(3, 1)\}$ and the point $p = (2, 1)$.

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