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Approximation Methods for Equilibrium Problems and Common Solution for a Finite Family of Inverse Strongly-Monotone Problems in Hilbert Spaces

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Abstract

The purpose of the paper is to investigate approximation methods for finding an element that is not only a solution of a equilibrium problem but also a common solution for a finite family of inverse strongly-monotone problems in Hilbert spaces.

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1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle ., . \rangle$ and $\|.\|$, respectively, let C be a nonempty closed (in the norm) and convex subset of H, and let F_0 be a bifunction from $C \times C$ to $\mathbf{R}$. The equilibrium problem for F_0 is to find $u^* \in C$ such that

$$F_0(u^*, v) \geq 0 \quad \forall v \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F_0)$. Assume that the bifunction F_0 satisfies the following set of standard properties.

Condition 1.1 The bifunction F is such that:

(A1) $F(u,u) = 0 \quad \forall u \in C$.
(A2) $F(u,v) + F(v,u) \leq 0 \quad \forall (u,v) \in C \times C$.
(A3) For every $u \in C$, $F(u,.) : C \to \mathbf{R}$ is lower semicontinuous and convex.
(A4) $\overline{\lim}_{t\to+0} F((1-t)u + tz, v) \leq F(u,v) \quad \forall (u,z,v) \in C \times C \times C$.

Let $T_i, i = 1, ..., N$ be a finite family of k_i-strictly pseudo-contractions from C into C with the nonempty set of fixed points $F(T_i)$ (i.e., $F(T_i) = \{x \in C : x = T_i x\}$). Assume that

$$\tilde{S} := \cap_{i=1}^N F(T_i) \cap EP(F_0) \neq \emptyset.$$

The problem of finding an element

$$u^* \in \tilde{S} \tag{1.2}$$

is studied intensively in [1]-[6], [9]-[11], and [13]-[25].

Recall that a mapping T in H is said to be a k-strictly pseudo-contraction in the terminology of Browder and Petryshyn [7] if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in D(T)$, the domain of definition of T, where I is the identity operator in H. Clearly, when $k = 0$, T is nonexpansive, i.e.,

$$\|T(x) - T(y)\| \leq \|x - y\|.$$

It means that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings.

In the case $T_i \equiv I$, (1.2) is the equilibrium problem (1.1) and shown in [5], [21] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems (see also [12]). For finding approximative solutions of (1.1) there exist several aproaches: the regularization approach in [9], [11], [13], [22], the gap-function approach in [13], [14], [16], and iterative procedure approach in [1]-[4], [6], [10], [17]-[20].

In the case $F_0 \equiv 0$ and $N = 1$, (1.2) is a problem of finding a fixed point for a k-strictly pseudo-contraction in C and studied in [15] where it is proved

Theorem 1.1. *Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a k-strict pseudo-contraction for some $0 \leq k < 1$ and assume that the fixed point set $F(T)$ of T is nonempty. Let $\{x_n\}$ be the sequence generated by the* *following (CQ) algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(k-\alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

Assume that the control sequence $\{\alpha_n\}$ *is chosen so that* $\alpha_n < 1$ *for all* n. *Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$, *the projection of* x_0 *onto* $F(T)$.

In the case $F_0 \equiv 0$ and $N > 1$, (1.2) is a problem of finding a common fixed point for a finite family of k_i-strictly pseudo-contraction T_i in C and studied in [25] where the following algorithm is constructed:

Let $x_0 \in C$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, and $\{u_n\}$ be a sequence in C. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_1 = \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\ x_2 = \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\ \ldots \\ x_N = \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\ x_{N+1} = \alpha_{N+1} x_N + \beta_{N+1} T_1 x_{N+1} + \gamma_{N+1} u_{N+1}, \\ \ldots \end{cases} \tag{1.3}$$

is called the implicit iteration process with mean errors for a family of strictly pseudo-contractions $\{T_i\}_{i=1}^N$.

The scheme (1.3) can be expessed in the compact form as

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n,$$

where $T_n = T_{n \bmod N}$. It is proved the following

Theorem 1.2. *Let* C *be a nonempty closed convex subset of* H. *Let* $\{T_i\}_{i=1}^N$ *be* N *strictly pseudo-contractive selfmaps of* C *such that* $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $x_0 \in C$ *and* $\{u_n\}$ *be a bounded sequence in* C, *let* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *be three sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\{\alpha_n\} + \{\beta_n\} + \{\gamma_n\} = 1,\ \forall n \geq 1$,

(ii) there exist constants σ_1, σ_2 *such that* $0 < \sigma_1 \leq \beta_n \leq \sigma_2 < 1, \forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the implicit iterative sequence $\{x_n\}$ *defined by (1.3) converges weakly to a common fixed point of the maps* $\{T_i\}_{i=1}^N$. *Moreover, in addition if there exists* $i_0 \in \{1, 2, ..., N\}$ *such that* T_{i_0} *is demicompact then* $\{x_n\}$ *converges strongly.*

In the case $F_0 \neq 0$ and $N = 1$, (1.2) is a problem of finding a fixed point for a k-strictly pseudo-contraction in C which is an equilibrium point for F, and studied in [24] where it is proved the following theorem.

Theorem 1.3. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n \end{cases}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\lim \inf_{n\to\infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

Set $A_i = I - T_i$. Obviously, A_i are λ_i inverse strongly-monotone, i.e.,

$$\langle A_i(x) - A_i(y), x - y\rangle \geq \lambda_i \|A_i(x) - A_i(y)\|^2 \quad \forall x, y \in D(A_i), \lambda_i = \frac{1 - k_i}{2}.$$

From now on, let $\{A_i\}_{i=1}^N$ be a finite family of λ_i inverse strongly-monotone operators in H with $C \subset \cap_{i=1}^N D(A_i)$ and $\lambda_i > 0, i = 1, ..., N$.

Set $S = \cap_{i=1}^N S_i$, where $S_i = \{x \in C : A_i(x) = 0\}$ is called the solution set of A_i in C.

Assume that $EP(F_0) \cap S \neq \emptyset$.

Our problem of investigation is to find an element

$$u^* \in EP(F_0) \cap S. \tag{1.4}$$

Because every nonexpansive mapping is 1/2 inverse strongly-monotone, the problem of finding an element $u^* \in C$ that is not only a solution of an inverse strongly-monotone problem but also a fixed point of a nonexpansive mapping is a particular case of (1.4) when $F_0 \equiv 0, N = 2$ and studied in [23] where it is proved the following theorem.

Theorem 1.4. *Let C be a nonempty closed convex subset of H. Let $\lambda > 0$. Let A be λ inverse strongly-monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$ where*

$VI(C,A)$ denotes the solution set of the following variational inequality: find $x_* \in C$ such that

$$\langle A(x_*), x - x_*\rangle \geq 0, \quad \forall x \in C.$$

Let $\{x_n\}$ be a sequence generated by

$$x_0 \in C,$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C(x_n - \lambda_n A(x_n)),$$

for every $n = 0, 1, ...,$ where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, 2\lambda)$ and $\{\alpha_n\} \subset (c,d)$ for some $c, d \in (0,1)$. Then, $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C,A)$, where

$$z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n.$$

In this paper, on the base of idea in [8] we present two methods of regularization which are the Tikhonov regularization and the regularization inertial proximal point algorithm for solving (1.4) where $F_0 \neq 0$ and $\{A_i\}_{i=1}^N$ are $\lambda_i(\lambda_i > 0)$ inverse strongly-monotone with that condition (A3) is replaced by

(A3') For every $u \in C$, $F_0(u,.) : C \to \mathbf{R}$ is weakly lower semicontinuous and convex.

The strong and weak convergences of any sequence are denoted by $\to$ and $\rightharpoonup$, respectively.

2. Main results.

We formulate the following facts in [5], [21] which are necessary in the proof of our results.

Proposition 2.1 (i) *If $F(.,v)$ is hemicontinuous for each $v \in C$ and F is monotone, i.e., satisfies (A2) in condition 1.1, then $U^* = V^*$, where*

U^* is the solution set of $F(u^*, v) \geq 0 \quad \forall v \in C$,

V^* is the solution set of $F(u, v^*) \leq 0 \quad \forall u \in C$,

and it is convex and closed.

(ii) If $F(.,v)$ is hemicontinuous for each $v \in C$ and F is strongly monotone, i.e., there exists a positive constant τ such that

$$F(u,v) + F(v,u) \leq -\tau\|u - v\|^2,$$

then U^* contains a unique element.

Lemma 2.1 *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be the sequences of positive numbers satisfying the conditions:*

(i) $a_{n+1} \leq (1-b_n)a_n + c_n, b_n < 1,$

(ii) $\sum_{n=0}^{\infty} b_n = +\infty, \quad \lim_{n\to+\infty} \frac{c_n}{b_n} = 0.$

Then, $\lim_{n\to+\infty} a_n = 0.$

Let S_A be a solution set of an inverse strongly-monotone operator A.

Lemma 2.2 *Let C_1 be a closed convex subset of C with the property $S_A \cap C_1 \neq \emptyset$. Then, the solution set of the following variational inequality*

$$\langle A(\tilde{y}), x - \tilde{y}\rangle \geq 0 \quad \forall x \in C_1, \tilde{y} \in C_1, \tag{2.1}$$

is coincided with $S_A \cap C_1$.

Proof. Obviously, every element in $S_A \cap C_1$ is a solution of (2.1). Let $\tilde{y}$ be an arbitrary solution of (2.1). We have to prove that $A(\tilde{y}) = 0$. Let $\tilde{x}$ be an element in $S_A \cap C_1$. Since $\tilde{x}$ is a zero element of the monotone operator A and $\tilde{y}$ is a solution of (2.1), then

$$0 = \langle A(\tilde{x}), \tilde{x} - \tilde{y}\rangle \geq \langle A(\tilde{y}), \tilde{x} - \tilde{y}\rangle \geq 0.$$

Hence, $\langle A(\tilde{y}), \tilde{x} - \tilde{y}\rangle = 0 = \langle A(\tilde{y}), \tilde{y} - \tilde{x}\rangle$. Consequently, $\langle A(\tilde{y}) - A(\tilde{x}), \tilde{y} - \tilde{x}\rangle = 0$. From the inverse strongly-monotone property of A it follows $A(\tilde{y}) = A(\tilde{x}) = 0$. It means that $\tilde{y} \in S_A \cap C_1$. Lemma is proved.

We construct the Tikhonov regularization solution u_α by solving the single equilibrium problem

$$\begin{aligned} &F_\alpha(u_\alpha, v) \geq 0 \quad \forall v \in C, u_\alpha \in C, \\ &F_\alpha(u, v) := \sum_{i=0}^{N} \alpha^{\mu_i} F_i(u, v) + \alpha\langle u, v - u\rangle, \alpha > 0, \\ &F_i(u, v) = \langle A_i(u), v - u\rangle, i = 1, ..., N, \\ &\mu_0 = 0 < \mu_i < \mu_{i+1} < 1, i = 2, ..., N-1, \end{aligned} \tag{2.2}$$

and α is the regularization parameter.

We have the following results.

Theorem 2.1. (i)*For each $\alpha > 0$, problem (2.2) has a unique solution u_α.*

(ii) $\lim_{\alpha \to +0} u_\alpha = u^*, u^* \in EP(F_0) \cap S, \|u^*\| \leq \|y\| \quad \forall y \in EP(F_0) \cap S.$

(iii)

$$\|u_\alpha - u_\beta\| \leq (\|u^*\| + dN)\frac{|\alpha - \beta|}{\alpha}, \quad \alpha, \beta > 0,$$

where d is a positive constant.

Proof. It is not difficult to verify that $F_i, i = 1, ..., N$, all are the bifunctions. Therefore, $F_\alpha(u, v)$ also is a bifunction, i.e. $F_\alpha(u, v)$ satisfies condition 1.1, and strongly monotone with constant $\alpha > 0$. Hence, (2.2) has a unique solution u_α for each $\alpha > 0$.

Now we shall prove that

$$\|u_\alpha\| \leq \|y\| \quad \forall y \in EP(F_0) \cap S. \tag{2.3}$$

Since $y \in EP(F_0) \cap S$, then $F_0(y, u_\alpha) \geq 0$ and $A_i(y) = 0, i = 1, ..., N$. Consequently, $F_i(y, u_\alpha) = 0, i = 1, ..., N$, and

$$\sum_{i=0}^{N} \alpha^{\mu_i} F_i(y, u_\alpha) \geq 0 \quad \forall y \in EP(F_0) \cap S.$$

This fact, u_α is the solution of (2.2) and the properties of F_i give

$$\langle u_\alpha, y - u_\alpha \rangle \geq 0 \quad \forall y \in EP(F_0) \cap S,$$

that implies (2.3). It means that $\{u_\alpha\}$ is bounded. Let $u_{\alpha_k} \rightharpoonup u^* \in H$, as $k \to +\infty$. Since C is closed in the norm and convex, then C is weak closed. Hence, $u^* \in C$. We prove that $u^* \in EP(F_0)$. From (A2) and (2.2) it follows

$$F_0(v, u_{\alpha_k}) + \sum_{i=1}^{N} \alpha_k^{\mu_i} F_i(v, u_{\alpha_k}) \leq \alpha_k \langle v, v - u_{\alpha_k} \rangle \quad \forall v \in C.$$

Using the property (A3') we obtain $F_0(v, u^*) \leq 0$ for any $v \in C$. By virtue of the proposition 2.1, we have $u^* \in EP(F_0)$. Now we show that $u^* \in S_i, i = 1, ..., N$. From (2.2), $F_0(y, u_{\alpha_k}) \geq 0$ for any $y \in EP(F_0)$, and the monotone property of F_0, i.e. $F_0(u_{\alpha_k}, y) + F_0(y, u_{\alpha_k}) \leq 0$, it implies that

$$\sum_{i=1}^{N} \alpha_k^{\mu_i} F_i(u_{\alpha_k}, y) + \alpha_k \langle u_{\alpha_k}, y - u_{\alpha_k} \rangle \geq 0 \quad \forall y \in EP(F_0).$$

Therefore,

$$F_1(y, u_{\alpha_k}) + \sum_{i=2}^{N} \alpha_k^{\mu_i - \mu_1} F_i(y, u_{\alpha_k}) \leq {\alpha_k}^{1-\mu_1} \langle y, y - u_{\alpha_k} \rangle \quad \forall y \in EP(F_0).$$

By tending $k \to \infty$, we have got

$$F_1(y, u^*) \leq 0 \quad \forall y \in EP(F_0)$$

that has the form

$$\langle A_1(y), y - u^* \rangle \geq 0 \quad \forall y \in EP(F_0).$$

The last inequality is equivalent to

$$\langle A_1(u^*), y - u^* \rangle \geq 0 \quad \forall y \in EP(F_0).$$

Since $EP(F_0) \cap F(T_1) \neq \emptyset$ and A_1 is an inverse strongly-monotone, from lemma 2.2 it follows $u^* \in S_1$.

Set $\tilde{S}_i = EP(F_0) \cap (\cap_{l=1}^{i} S_l)$. Then, $\tilde{S}_i$ is also closed convex, and $\tilde{S}_i \neq \emptyset$.

Now, suppose that we have proved that $u^* \in \tilde{S}_i$, and need to show that u^* belongs to S_{i+1}. Again, by virtue of (2.2) for $y \in \tilde{S}_i$ we can write

$$F_{i+1}(y, u_{\alpha_k}) + \sum_{l=i+2}^{N} \alpha_k^{\mu_l - \mu_{i+1}} F_l(y, u_{\alpha_k}) \leq \alpha_k^{1-\mu_{i+1}} \langle y, y - u_{\alpha_k} \rangle \quad \forall y \in \tilde{S}_i.$$

After passing $k \to \infty$, we obtain

$$F_{i+1}(y, u^*) \leq 0 \quad \forall y \in \tilde{S}_i.$$

By virtue of $\tilde{S}_i \cap S_{i+1} \neq \emptyset$, u^* also is an element of S_{i+1}, i.e., $F_{i+1}(y, u^*) \leq 0 \quad \forall y \in \tilde{S}_i$. Inequality (2.3) and the weak convergence of $\{u_{\alpha_k}\}$ to $u^* \in EP(F_0) \cap S$, which is a closed convex subset in H, give the strong convergence of $\{u_{\alpha_k}\}$ to $u^* : \|u^*\| \leq \|y\| \quad \forall y \in EP(F_0) \cap S$.

From (2.2) and the properties of $F_i(u, v)$, for each $\alpha, \beta > 0$ it follows

$$\sum_{i=0}^{N} (\alpha^{\mu_i} - \beta^{\mu_i}) F_i(u_\alpha, u_\beta) + \alpha \langle u_\alpha, u_\beta - u_\alpha \rangle + \beta \langle u_\beta, u_\alpha - u_\beta \rangle \geq 0$$

or

$$\|u_\alpha - u_\beta\| \leq \frac{|\alpha - \beta|}{\alpha} \|u_\beta\| + \frac{1}{\alpha} \sum_{i=1}^{N} |\alpha^{\mu_i} - \beta^{\mu_i}| |F_i(u_\alpha, u_\beta)|,$$

because $\mu_0 = 0$. All $F_i, i = 1, ..., N$, are bounded, because the operators A_i all are Lipschitzian with Lipschitz constants $L_i = 1/\lambda_i$. Using (2.3), the boundedness of F_i and the Lagrange's mean-value theorem for the function $\alpha(t) = t^{-\mu}, 0 < \mu < 1, t \in [1, +\infty)$, on $[\alpha, \beta]$ if $\alpha < \beta$ or $[\beta, \alpha]$ if $\beta < \alpha$ we have conclusion (iii). Theorem is proved now.

Remark. Obviously, if $u_{\alpha_k} \to \tilde{u}$, where u_{α_k} is the solution of (2.2) with $\alpha = \alpha_k \to 0$, as $k \to +\infty$, then $EP(F_0) \cap S \neq \emptyset$.

Further, we consider the regularization inertial proximal point algorithm where z_{n+1} is defined by

$$\tilde{c}_n \left(\sum_{i=0}^{N} \alpha_n^{\mu_i} F_i(z_{n+1}, v) + \alpha_n \langle z_{n+1}, v - z_{n+1} \rangle \right) + \langle z_{n+1} - z_n, v - z_{n+1} \rangle \\ - \gamma_n \langle z_n - z_{n-1}, v - z_{n+1} \rangle \geq 0 \quad \forall v \in C, z_0, z_1 \in C, \tag{2.4}$$

and $\{\tilde{c}_n\}$ and $\{\gamma_n\}$ are the sequences of positive numbers. Note that in the case $N = 0$ algorithm (2.4) is considered in [18] without the regularized term $\alpha_n \langle z_{n+1}, v - z_{n+1} \rangle$, and the obtained result only is the weak convergence of the sequence $\{z_n\}$ under some condition. By virtue of this term we shall obtain a stronger result.

It is not difficult to verify that the bifunction

$$\tilde{c}_n \left(\sum_{i=0}^{N} \alpha_n^{\mu_i} F_i(u, v) + \alpha_n \langle u, v - u \rangle \right) + \langle u - z_n, v - u \rangle - \gamma_n \langle y_n, v - u \rangle,$$

where $y_n = z_n - z_{n-1}$, is strongly monotone with constant $\tilde{c}_n \alpha_n$. Therefore, (2.4) possesses a unique solution z_{n+1} for each n.

Theorem 2.2 *Assume that the parameters $\tilde{c}_n, \gamma_n$ and α_n are chosen such that:*
(i) $0 < c_0 < \tilde{c}_n < C_0, 0 \le \gamma_n < \gamma_0,$
(ii) $\sum_{n=1}^{\infty} b_n = +\infty, b_n = \tilde{c}_n\alpha_n/(1+\tilde{c}_n\alpha_n),$
(iii) $\sum_{n=1}^{\infty} \gamma_n b_n^{-1}\|z_n - z_{n-1}\| < +\infty,$
(iv) $\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n b_n} = 0.$
Then, the sequence $\{z_n\}$ defined by (2.4) converges strongly to the element u^*, as $n \to +\infty$.
Proof. Denote by u_n and u_{n+1} the solutions of (2.2) with $\alpha = \alpha_n$ and $\beta = \alpha_{n+1}$, respectively. Then, we have the following inequality

$$\|u_{n+1} - u_n\| \le (\|u^*\| + dN)\frac{\alpha_n - \alpha_{n+1}}{\alpha_n},$$
$$d = \max_{1\le i\le N}\{\frac{4\|u^*\|^2}{\lambda_i}\}.$$

On the other hand, (2.4) and (2.2) can be rewritten in the equivalent forms

$$\tau_n \sum_{j=0}^{N} \alpha_n^{\mu_i} F_i(z_{n+1}, v) + \langle z_{n+1}, v - z_{n+1}\rangle \ge \beta_n\langle z_n, v - z_{n+1}\rangle$$
$$+ \beta_n\gamma_n\langle z_n - z_{n-1}, v - z_{n+1}\rangle \quad \forall v \in C,$$

$$\tau_n \sum_{i=0}^{N} \alpha_n^{\mu_i} F_i(u_n, v) + \langle u_n, v - u_n\rangle \ge \beta_n\langle u_n, v - u_n\rangle, \quad \forall v \in C,$$

respectively, where $\tau_n = \tilde{c}_n\beta_n, \beta_n = 1/(1+\tilde{c}_n\alpha_n)$. Replacing $v = u_n$ and $v = z_{n+1}$ in the last two inequalities, respectively, and then summarizing the results, we obtain the inequality

$$\langle z_{n+1} - u_n, u_n - z_{n+1}\rangle \ge \beta_n\langle z_n - u_n, u_n - z_{n+1}\rangle$$
$$+ \beta_n\gamma_n\langle z_n - z_{n-1}, u_n - z_{n+1}\rangle.$$

Consequently,

$$\|z_{n+1} - u_n\| \le \beta_n\|z_n - u_n\| + \beta_n\gamma_n\|z_n - z_{n-1}\|.$$

Hence,

$$\begin{aligned}\|z_{n+1} - u_{n+1}\| &\le \|z_{n+1} - u_n\| + \|u_{n+1} - u_n\| \\ &\le \beta_n\|z_n - u_n\| + \beta_n\gamma_n\|z_n - z_{n-1}\| \\ &\quad + (\|u^*\| + dN)\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} \\ &\le (1 - b_n)\|z_n - u_n\| + c_n,\end{aligned}$$
$$c_n = \beta_n\gamma_n\|z_n - z_{n-1}\| + (\|u^*\| + dN)\frac{\alpha_n - \alpha_{n+1}}{\alpha_n}$$

Since the serie in (iii) is convergent, then $\beta_n\gamma_n\|z_n - z_{n-1}\|b_n^{-1} \le \gamma_n\|z_n - z_{n-1}\|b_n^{-1} \to 0$, as $n \to \infty$. This fact and (iv) follow $\lim_{n\to\infty} c_n b_n^{-1} = 0$. By using the above lemma with $a_n = \|z_n - u_n\|$ we have

$$\lim_{n\to\infty} \|z_n - u_n\| = 0.$$

Since $u_n \to u^*$, then $z_n \to u^*$, as $n \to \infty$. Theorem is proved.
Remark The sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ which are defined by

$$\alpha_n = (1+n)^{-p}, 0 < p < 1/2,$$
$$\gamma_n = (1+n)^{-\tau}\frac{\|z_n - z_{n-1}\|}{1 + \|z_n - z_{n-1}\|^2},$$

with $\tau > 1 + p$ satisfy all conditions in theorem 2.2.

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