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Some Results Concerning Quasi-continuity and Fragmentability

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Abstract

We shall show that every quasi-continuous mapping $g : Z \longrightarrow X$ of a topological space Z into a topological space X which has countable base can be discontinuous only on a set of first category. Also it is shown that every quasi-continuous map $g : Z \longrightarrow X$ of a Baire space Z into a fragmentable compact space X in a dense G_δ-subset is continuous.

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1 Introduction

In the paper [1] Kempisty introduced a notion similar to continuity for real-valued functions defined in **R**. For general topological spaces this notion can be given the following equivalent formulation.

Definition 1.1. *The mapping $g : Z \longrightarrow X$ between the topological spaces Z and X is said to be quasi-continuous at $z_0 \in Z$ if for every neighborhood U of $g(x_0)$, there exists some open set $V \subset Z$ such that*
(a) $z_0 \in \overline{V}$ *(the closure of V in Z), and*
(b) $g(V) \subset U$.

The mapping g is called quasi-continuous if it is quasi-continuous at each point of Z.

Remark 1.2. Continuity implies quasi-continuity, of course, but not conversely; consider, in fact,
Example 1.3. Let $Z = X = [0, 1]$. Let $g : Z \longrightarrow X$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{if } 1/2 < x \leq 1 \end{cases}$$

The example below shows that a quasi-continuous mapping need not be continuous at any point.

Example 1.4. Take $Z = [0, 1]$ with the usual topology, $X = [0, 1]$ with the Sorgenfrey topology and the identity mapping $g : Z \longrightarrow X$. The map g is quasi-continuous but nowhere continuous. We left the easy details to the reader.

Nevertheless, under some mild requirements imposed on the spaces Z and X, every quasi-continuous map becomes continuous at many points of the space Z (see Theorem 3.2 and Corollary 3.5).

2 Quasi-open sets

Definition 2.1. *A set A in a topological space Z will be called quasi-open (written q.o.) if there exists an open set O such that $O \subset A \subset clO$ where cl denotes the closure operator in Z. Q.O.(Z) will denote the quasi-open sets in Z.*

Theorem 2.2. *Let Z be a topological space. Then the following are true:*
(1) A subset A in Z is q.o. if and only if $A \subset clIntA$, where Int denotes the interior operator.
(2) Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of q.o. sets in Z. Then $\cup_{\alpha\in\Delta}A_\alpha$ is q.o.
(3) Let A be q.o. in Z and $A \subset B \subset clA$. Then B is q.o.
(4) If O is open in Z, then O is q.o. in Z. The converse is clearly false.

proof. ($\Rightarrow$) Let A be q.o. Then $O \subset A \subset clO$ for some open set O. But $O \subset IntA$ and thus $clO \subset clIntA$. Hence $A \subset clO \subset clIntA$.
($\Leftarrow$) Let $A \subset clIntA$. Then for $O = IntA$, we have $O \subset A \subset clO$.

(2) For each $\alpha \in \Delta$, we have an O_α such that $O_\alpha \subset A_\alpha \subset clO_\alpha$. Then $\cup_{\alpha\in\Delta}O_\alpha \subset \cup_{\alpha\in\Delta}A_\alpha \subset \cup_{\alpha\in\Delta}clO_\alpha \subset cl \cup_{\alpha\in\Delta} O_\alpha$. Hence let $O = \cup_{\alpha\in\Delta}O_\alpha$.

(3) There exists an open set O such that $O \subset A \subset clO$. Then $O \subset B$, But $clA \subset clO$ and thus $B \subset clO$. Hence $O \subset B \subset clO$ and B is q.o.

(4) It is clear.□

Corollary 2.2. *Let τ be the class of open sets in Z. Then*
(1) $\tau \subset Q.O.(Z)$.
(2) for $A \in Q.O.(Z)$ and $A \subset B \subset ciA$, then $B \in Q.O.(Z)$.
proof. This follows from Theorem 2.1.□

Theorem 2.3. *Let $\mathcal{B} = \{B_\alpha\}$ be a collection of sets in Z such that*
(1) $\tau \subset \mathcal{B}$,
(2) if $B \in \mathcal{B}$ and $B \subset D \subset clB$, then $D \in \mathcal{B}$.
Then $Q.O.(Z) \subset \mathcal{B}$. Thus $Q.O.(Z)$ is the smallest class of sets in Z satisfying (1) and (2).

proof. Let $A \in Q.O.(Z)$. Then $O \subset A \subset clO$ for some $O \in \tau$. Therefore $O \in \mathcal{B}$ by (1) and thus $A \in \mathcal{B}$ by (2).□

Theorem 2.4. *Let $A \subset W \subset Z$ where Z is a topological space and W a subspace. Let $A \in Q.O.(Z)$. Then $A \in Q.O.(W)$.*

proof. $O \subset A \subset cl_Z O$ where O is open in Z and cl_Z denotes the closure operator in Z. Now $O \subset W$ and thus $O = O \cap W \subset A \cap W \subset W \cap cl_Z O$ or $O \subset A \subset cl_Z O$. Since $O = O \cap W$, O is open in W and the theorem is proved. □

The converse of Theorem 2.4 is false, as shown by

Example 2.5. Let Z be the space of reals and $W = A = \{0\}$. Then A is open in W and hence $A \in Q.O.(W)$. But $A \notin Q.O.(Z)$.

Theorem 2.6. *Let $A \in Q.O.(Z)$ where Z is a topological space. Then $A = O \cup B$ where*
(1) $O \in \tau$,
(2) $O \cap B = \emptyset$ and
(3) B is nowhere dense.
proof. $O \subset A \subset cl\, O$ for some O open in Z. But $A = O \cup (A \setminus O)$. Let $B = A \setminus O$. Then $B \subset cl\, O \setminus O$ and is nowhere dense by Theorem 2.1. Then

$A = O \cup B$, and (1) and (2) immediately follow. $\square$

The converse of Theorem 2.6 is false, as shown by

Example 2.7. Let Z be the space of reals and $A = \{x : 0 < x < 1\} \cup \{2\}$. Then $A \notin Q.O.(Z)$ although (1), (2) and (3) in Theorem 2.6 hold.

The converse of Theorem 2.6 is false even when connectedness is imposed upon A, as shown by

Example 2.8. Let Z be the plane and
$A = \{(x, y) : 0 < x < 1 \ \textit{and} \ 0 < y < 1\} \cup \{(x, y) : 1 \leq x \leq 2\}$.
It is clear that $A \ni Q.O.(Z)$ although again (1), (2) and (3) in Theorem 2.9 are satisfied.

Remark 2.9. It is not true that components of quasi-open sets are quasi-open, as shown by

Example 2.10. Let Z be the space of reals and

$$A = \{0\} \cup (1/2, 1) \cup (1/4, 1/2) \cup \cdots \cup (1/2^{n+1}, 1/2^n) \cup \cdots$$

Then A is quasi-open and $\{0\}$ is a component of A which is not q.o. in Z.

In general the complement of a q.o. set is not q.o. nor is the intersection of two q.o. sets q.o.

The Theorem below shows that a continuous map which is also open sends quasi-open sets to quasi-open sets.

Theorem 2.11. *Let $g : Z \longrightarrow X$ be continuous and open where Z and X are topological spaces. If $A \in Q.O.(Z)$ then $f(A) \in Q.O.(X)$.*

proof. Let $A = O \cup B$ where O is open and $B \subset clO \setminus O$. Then $f(O) \subset f(A) = f(O) \cup f(B) \subset f(O) \cup f(clO) \subset f(O) \cup clf(O) = clf(O)$. Hence, $F(O) \subset f(A) \subset clf(O)$ and $f(O)$ is open in X since $g : Z \longrightarrow X$ is open. $\square$

If "open" is removed from Theorem 2.21, then the Theorem is in general false, as shown by

Example 2.12. Let Z and X both be the space of reals and $g : Z \longrightarrow X$ be the constant map 1. Then Z is quasi-open in Z but $f(Z) = \{1\}$ is not quasi-open in X.

Definition 2.13. *Let Z be a topological space and $\mathcal{B} = \{B_\alpha\}$ a collection of subsets. Then $Int\,\mathcal{B}$ will denote $\{Int\,B_\alpha\}$.*

Lemma 2.14. *Let τ be the class of open sets in the topological space Z. Then $\tau = Int\,Q.O.(Z)$.*

proof. Let $O \in \tau$. Then $O \in Q.O.(Z)$ and since $O = IntO$, $O \in Int\,Q.O.(Z)$. $\Box$
Conversely let $O \in Q.O.(Z)$. Then $O = IntA$ for some $A \in Q.O.(Z)$ and thus $O \in \tau$. $\Box$

Example 2.15. Let Z be the set of reals and τ the topology generated by sets of the form (x, Y) where x is less than y. Let τ^* be the topology generated by the sets of the form $[x, y)$ where again x is less than y. Then $\tau \subset \tau^*$, but $Q.O.(Z, \tau) \not\subset Q.O.(Z, \tau^*)$ since $(x, y] \in Q.O.(Z, \tau)$, but $(x, y] \notin Q.O.(Z, \tau^*)$.

3 Quasi-continuity revisited

Theorem 3.1. *The mapping $g : Z \longrightarrow X$ between the topological spaces Z and X is quasi-continuous if and only if $g^{-1}(O) \in Q.O.(X)$ for every open subset O of Z.*

proof. ($\Rightarrow$) Let O be open in Z and $p \in g^{-1}(O)$. Then $g(p) \in O$ and thus there exists an $A_p \in Q.O.(Z)$ such that $p \in A_p$ and $f(A_p) \subset O$. Then $p \in A_p \subset g^{-1}(O)$ and $g^{-1}(O) = \cup_{p \in g^{-1}(O)} A_p$. Then by Theorem 2.1, $g^{-1}(O) \in Q.O.(Z)$.
($\Leftarrow$) Let $g(p) \in O$. Then $p \in g^{-1}(O) \in Q.O.(Z)$ since $g : Z \longrightarrow X$ is quasi-continuous. Let $A = g^{-1}(O)$. Then $p \in A$ and $g(A) \in O$.$\Box$

Example 1.4 shows that a quasi-continuous map can be discontinuous everywhere. Also a quasi-continuous map from a topological space which has countable base into another topological space could only be discontinuous in a set of first category.

Theorem 3.2. *Let $g : Z \longrightarrow X$ be quasi-continuous and X a topological space which has countable base. Let D be the set of points of discontinuity of g. Then D is of first category.*

proof. Let $p \in D$. Then for any integer n, there exists an open set O_{n_p} in the countable open basis for X such that $g(U) \subset O_{n_p}$ for every open neighborhood U of p. Now there exists an $A_{n_p} \in Q.O.(Z)$ such that $p \in A_{n_p}$ and $g(A_{n_p}) \subset U_{n_p}$ by discontinuity of g at p. But $A_{n_p} = U_{n_p} \cup B_{n_p}$ where $B_{n_p} \subset clU_{n_p} \setminus U_{n_p}$. Hence $p \notin U_{n_p}$ and thus '$p \in B_{n_p}$, a nowhere dense set. It follows then that $D \subset \cup_{p\in D} B_{n_p}$ and since $\cup_{p\in D} B_{n_p}$ is of first category, it follows that D is of first category.□

Definition 3.3. *Let X be a topological space and ρ some metric defined on X. The space X is said to be fragmented by the metric ρ if for every $\epsilon > 0$ and every subset $A \subset X$, there exists a non-empty relatively open subset $B \subset A$ with $\rho - diam(B) \leq \epsilon$.*
In such a case the space X is called fragmentable.

The proof of the next result shows some techniques associated with quasi-continuity of mappings and fragmentability of spaces.

Theorem 3.4. *Let Z be a Baire space and $g : Z \longrightarrow X$ a quasi-continuous map from a topological space which is fragmented by some metric ρ. Then there exists a G_δ-subset $C \subset Z$ at the point of which $g : Z \longrightarrow (X, \rho)$ is continuous. In particular if the topology generated by the metric ρ contains the topology of the space X, then $g : Z \longrightarrow X$ is continuous at every point of the set C.*

proof. Consider for every $n = 1, 2, \cdots$ the set $V_n = \cup\{V : V \text{ } open \text{ } in \text{ } Z \text{ } and \text{ } \rho - diam(g(V)) \leq n^{-1}\}$. The set V_n is open in Z.
Suppose W is a non-empty open subset of Z. Consider the set $A = g(W)$ by fragmentability of X, there is some relatively open subset $B = A \cap U = g(W) \cap U$ where U is open in X such that $\rho - diam(B) \leq n^{-1}$. Quasi-continuity of g implies that there exists some non-empty open $V \subset W$ with $g(V) \subset U \cap g(W) = B$ this shows that $\emptyset \neq V \subset V_n \cap W$. Hence V_n is dense in Z. Obviously, at each point of $C = \cap_{n\geq 1} V_n$ the map g is ρ-continuous. □

Note that according to a result of Ribarska [3]-[4], if the space X is compact and fragmentable, then it is also fragmentable by some metric that majorizes the topology of X. I.e the metric topology generated by the new metric contains the topology of the compact space X.

Corollary 3.5. *Let Z be a Baire space and $g : Z \longrightarrow X$ a quasi continuous map from Z into the fragmentable compact space X. Then there exists a dense*

G_δ-subset $C \subset Z$ at the point of which $g : Z \longrightarrow X$ is continuous.

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