Applied Mathematical Sciences, Vol. 2, 2008, no. 16, 775 - 780

b-General Orthogonality in 2-Normed Spaces

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Abstract

In this paper, we investigate some properties of the b-General orthogonality in 2-normed spaces, and obtain some results on b-General orthogonality in 2-normed spaces similar to b-orthogonality of 2-normed spaces. In this paper we shall consider the relation between this concept in 2-smooth spaces and sense b-Brikhoff orthogonality.

Mathematics Subject Classification: 41A65, 41A15

Keywords: b-General orthogonality, 2-smooth spaces, b-Best approximation

1. Introduction

The concept of linear 2-normed spaces has been investigated by S. Gähler and has been developed extensively in different subjects by many authors (see [1-6]).

Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:
a) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.
b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
c) $\|\lambda x, y\| = |\lambda|\|x, y\|$ for all $\lambda \in \mathbf{R}$ and all $x, y \in X$.

d) $\|x+y,z\| \leq \|x,z\| + \|y,z\|$ for all $x,y,z \in X$.
Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. Some of the basic properties of 2-norms, that they are non-negative and $\|x, y+\alpha x\| = \|x,y\|$ for all $x,y \in X$ and all $\alpha \in \mathbf{R}$.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x,b\|$, $x \in X$, is a seminorm and the family $P = \{p_b : \ b \in X\}$ of seminorms generates a locally convex topology on X.

Let $(X, \|.,.\|)$ be a 2-normed space and let W_1 and W_2 be two subspaces of X. A map $f : W_1 \times W_2 \to \mathbf{R}$ is called a bilinear 2-functional on $W_1 \times W_2$ whenever for all $x_1, x_2 \in W_1, y_1, y_2 \in W_2$ and all $\lambda_1, \lambda_2 \in \mathbf{R}$;
(i) $f(x_1+x_2, y_1+y_2) = f(x_1,y_1) + f(x_1,y_2) + f(x_2,y_1) + f(x_2,y_2)$,
(ii) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 f(x_1, y_1)$.

A bilinear 2-functional $f : W_1 \times W_2 \to \mathbf{R}$ is called bounded if there exists a non-negative real number M (called a Lipschitz constant for f) such that $|f(x,y)| \leq M\|x,y\|$ for all $x \in W_1$ and all $y \in W_2$. Also, the norm of a bilinear 2-functional f is defined by

$$\|f\| = \inf\{M \geq 0 : \ M \ is \ a \ Lipschitz \ constant \ for \ f\}.$$

It is known that ([4])

$$\begin{aligned} \|f\| &= \sup\{|f(x,y)| : \ (x,y) \in W_1 \times W_2, \ \|x,y\| \leq 1\} \\ &= \sup\{|f(x,y)| : \ (x,y) \in W_1 \times W_2, \ \|x,y\| = 1\} \\ &= \sup\{|f(x,y)|/\|x,y\| : \ (x,y) \in W_1 \times W_2, \ \|x,y\| > 0\}. \end{aligned}$$

For a 2-normed space $(X, \|.,.\|)$ and $0 \neq b \in X$, we denote by X_b^* the Banach space of all bounded bilinear 2-functionals on $X \times < b >$, where $< b >$ be the subspace of X generated by b.

Let $(X, \|.,.\|)$ be a 2-normed space, $x,y \in X$. If there exists $b \in X$ such that $\|x,b\| \neq 0$ and $\|x,b\| \leq \|x+\alpha y, b\|$ for all scalar $\alpha \in \mathbf{R}$. Then x is b-Brikhoff orthogonal to y and denoted by $x \perp_{BG}^b y$.

If W_1 and W_2 are subsets of X, if there exists $b \in X$ such that for all $y_1 \in W_1$, $y_2 \in W_2$, $y_1 \perp^b y_2$, then we say that $W_1 \perp_{BG}^b W_2$.

Let $(X, \|.,.\|)$ be a 2-normed space, W be a linear subspace of X and $b \in X$. $w_0 \in W$ is b-best approximation for $x \in X$, if $x - w_0 \perp_{BG}^b W$. The set of all b-best approximations of x in W is denoted by $P_W^b(x)$. Also W is called b-proximinal if for every $x \in X \backslash (W + < b >)$, there exists $w_0 \in W$ such that $w_0 \in P_W^b(x)$. (see [10])

Let X be a 2-normed linear space. We say that X is a 2-smooth if for any $x \neq 0$ and every $b \in X$ such that $\|x, b\| \neq 0$, there is a unique linear functional Λ_b such that

$$\Lambda_b(x) = \|x, b\|, \quad \|\Lambda_b\| = 1$$

The following basic lemmas is important in the proof of main results.

Proposition 1.2 (3]) *Let $(X, \|.,.\|)$ be a 2-normed space,W be a subspace of X, $b \in X$ and let $< b >$ be the subspace of X generated by b. If $x_0 \in X$ is such that*

$$\delta = \inf\{\|x_0 - w, b\| : \ w \in W\} > 0,$$

then there exists a bounded bilinear functional $F : \ X\times < b > \rightarrow \boldsymbol{R}$ such that $F|_W \times < b >= 0$, $F(x_0, b) = 1$ and $\|F\| = \frac{1}{\delta}$.

Lemma 1.2. *Let X be a 2-normed linear space, $b \in X$, $y \in X$ and $x \in X\backslash(< b >)$. Then the following statements are equivalent:*
1) $x \perp^b y$.
2) There exists X^*_b such that $f(y, b) = 0$, $f(x, b) = \|x, b\|$ and $\|f\| = 1$.

Lemma 1.3. *Let X be a 2-normed linear space. Let W be a linear subspace of X, $b \in X$ and $F \subseteq X\backslash(W+ < b >)$. Then the following statements are equivalent:*
1) $F \perp^b W$.
2) There exists $f \in X^*_b$ such that $f|_{W\times<b>} = 0$, $\|f\| = 1$ and $f(x, b) = \|x, b\|$ for all $x \in F$.

Let X be a linear space of dimension greater then 1 over the filed $\mathbf{K} = \mathbf{R}$ of real numbers or the filed $\mathbf{K} = \mathbf{C}$ of complex numbers. Suppose that $(.,.|.)$ is a $\mathbf{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:
a) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent;
b) $(x, x|z) = (z, z|x)$;
c) $(y, x|z) = \overline{(x, y|z)}$;
d) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbf{K}$;
e) $(x + x', y|z) = (x, y|z) + (x', y|z)$.
$(.,.|.)$ is called a 2-inner product on X and $(X, (.,.|.))$ is called a 2-inner product space. some basic properties of 2-inner products $(.,.|.)$ can be immediately obtained in [1-3].

Let $(X, (.,.|.))$ be a 2-inner product space. We can define a 2-norm on $X \times X$ by

$$\|x, y\| = \sqrt{(x, x|y)}.$$

we shall consider general orthogonality in the Banach spaces.

Definition 1.4. *Let X be a 2-normed linear space and $x, y, b \in X$. x is called b-general orthogonal to y and write $x \perp_G^b y$, if and only if there exists a unique $\phi_x \in X_b^*$ such that $\phi_x(x,b) = \|x,b\|^2$, $\|\phi_x\| = \|x,b\|$ and $\phi_(y,b) = 0$.*

2. Main Results

In this section we state and prove some characterizations of the b-general orthogonality in 2-normed spaces.

Theorem 2.1. *Let X be a 2-normed linear space. Then the following statements are true:*
a) *For all $x \in X$ and all $\alpha > 0$, $\phi_{\alpha x} = \alpha \phi_x$.*
b) *For all $x, y \in X$ and all $\alpha > 0$, if $x \perp_G^b y$, then $\alpha x \perp_G^b y$.*
c) *For all $x \in X$, if $x \perp_G^b x$ then $x = 0$.*
d) *For all $x, y \in X$, if $x \perp_G^b y$ and $x \neq 0$, then $< x > \cap < y >= \{0\}$.*
e) *For all $x \in X$, $0 \perp_G^b x$ and $x \perp_G^b 0$.*

Proof. (a). Suppose $x \in X$ and $\alpha > 0$. Then

$$\alpha\phi_x(\alpha x) = \alpha^2\|x,b\|^2 = \|\alpha x, b\|^2 \ , \|\alpha\phi_x, b\| = \alpha\|\phi_x, b\| = \alpha\|x,b\| = \|\alpha x, b\|,$$

also $\alpha\phi_x(y) = 0$. By uniqueness of $\phi_{\alpha x}$ we have $\phi_{\alpha x} = \alpha\phi_x$.

(b). Proof is a conclusion of (a).

(c). For all $x \in X$, if $x \perp^G x$. Then $\phi_x(x,b) = 0$ and $\phi_x(x,b) = \|x,b\|^2$. Therefore $x = 0$.

(d). If $z \in < x > \cap < y >$, then for scales c_1, c_2, $z = c_1 x = c_2 y$. Hence $\phi_x(z,b) = 0$, it follows that $\phi_x(c_1 x, b) = 0$. Therefore $c_1 = 0$ and $z = 0$.

(e). It is trivial. ■

Let X be a 2-normed space. The element $x \in B$ is called 2-normal element if there exists only one $f \in X_b^*$ such that $f(x) = \|x,b\|$ and $\|f\| = 1$.

Theorem 2.2. *Let X be a 2-normed space. Then the following statements are true:*
a) *If $x, y \in X$, $x \perp_G^b y$, then $x \perp_{BG}^b y$.*
b) *If $x \neq 0 \in B$ is a 2-normal element, $y \in B$ and $x \perp_{BG}^b y$, then $x \perp_G^b y$.*

Proof. (a). Suppose $x, y \in X$ and $x \perp_G^b y$ then

$$\begin{aligned} \|x,b\|^2 &= \phi_x(x,b) \\ &= \phi_x(x + \alpha y, b) \\ &\leq \|\phi_x\|\|x + \alpha y, b\| \\ &= \|x,b\|\|x + \alpha y, b\|. \end{aligned}$$

Therefore $\|x,b\| \leq \|x + \alpha y, b\|$, that is $x \perp_{BG}^b y$.

(b). We know that if $x\perp^b_{BG}y$ and $\alpha > 0$ then $\alpha x\perp^b_{BG}y$. Therefore $Z = \frac{x}{\|x\|}\perp^b_{BG}y$. Since x is normal by Lemma 1.2, there exists a unique $\phi_Z \in X^*_b$ such that $\phi_Z(X) = 1$, $\|\phi_Z\| = 1$ and $\phi_Z(y, b) = 0$. From Theorem 2.1, $\phi_Z = \frac{1}{\|x\|}\phi_x$. Therefore there exists a unique $\phi_x \in X^*_b$ such that $\phi_x(x) = \|x, b\|^2$, $\|\phi_x\| = \|x, b\|$ and $\phi_(y, b) = 0$. It follows that $x\perp^b_G y$. ■

We know that every 2-inner product space $(X, < ., .|. >)$ is a 2-smooth spaces. Therefore every element $x \in X$ is normal, Hence we have

Corollary 2.3. *Let $(X, < ., .|. >)$ be a 2-smooth spaces, $x, y, b \in X$. If $x\perp^b_G y$ then $< x, y|b >= 0$.* ■

Definition 2.4. *Let X be a 2-normed space and $b \in X$. The b-general orthogonality is called $b - G$-additivity, if $y\perp^b_G x$ and $z\perp^b_G x$, then $y + z\perp^b_G x$.*

Definition 2.5. *Let X be a 2-normed space, $M \subseteq X$ and $x, b \in X$. Then we say that x is b-general orthogonal to M and write $x\perp^b_G M$ if and only if there exists a unique $\phi_x \in X^*_b$ such that $\phi_x(x, b) = \|x, b\|^2$, $\|\phi_x\| = \|x, b\|$ and for all $y \in M$ $\phi_(y, b) = 0$. The element $y_0 \in M$ is $b - G$-best approximation of $x \in X$ if and only if $x - y_0\perp^b_G M$.*

Corollary 2.6. *Let X be a 2-normed space, $x, b \in X$. If $y_0 \in M$ is $b - G$-best approximation of x. Then y_0 is a b-best approximation of x.*

Theorem 2.7. *Let X be a 2-normed space, $x, b \in X$ and $M \subseteq X$. If the b-general orthogonality is $b - G$-additivity, Then there exist a unique $y_0 \in M$ such that $x - y_0\perp^b_G M$.*

Proof. *Suppose $y_1, y_2 \in M$ such that for all $i = 1, 2$, $x - y_i\perp^b_G M$. Therefore for all $y \in M$ and all $i = 1, 2$, $x - y_i\perp^b_G y$. Since b-orthogonality is $b - G$-additivity. It follows that $y_2 - y_1 = (x - y_1) - (x - y_2)\perp^b_G y$. Put $y = y_1 - y_2$, then $y_1 - y_2\perp^b_G y_1 - y_2$. From Theorem 2.1, $y_1 = y_2$.* ■

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Received: May 30, 2007