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Formulas for the Number of Spanning Trees in a Fan

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Abstract

Let P_n be a simple path on n vertices. An *n-fan* is a simple graph G formed from a path P_n by adding a vertex adjacent to every vertex of P_n. In this work we denote n-fan by F_{n+1} and derive the explicit formula for $t(F_{n+1})$ the number of spanning trees in F_{n+1} to be $t(F_{n+1}) = 2\frac{((3-\sqrt{5})/2)^{n+1}-((3+\sqrt{5})/2)^{n-1}}{5-3\sqrt{5}}$. In addition, we show that $t(F_{n+1}) = F\_\{2n\}$, where $F\_\{i\}$ represents i'th Fibonacci number.

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1 Introduction

In this paper we derive two simple formulas for the number of spanning trees of a special family of graphs called *n-fans* [5]. Let P_n be a simple path defined on n vertices. An undirected simple graph F on $n + 1$ vertices is defined as an *n-fan* when it is obtained from P_n by adding an additional vertex adjacent to every vertex of P_n. In this work we denote n-fan by F_{n+1}. The n-fan is illustrated in Figure 1.

The number of spanning trees of G, denoted by $t(G)$, is the total number of distinct spanning subgraphs of G that are trees. A classic result of Kirchhoff [4] can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = v_1, v_2, ..., v_n$. To state the result, we define the $n \times n$ characteristic matrix $A = [a_{ij}]$ as follows: (i) $a_{ij} = -1$ if v_i and v_j are adjacent and $i \neq j$, (ii) a_{ij} equals the degree of vertex v_i if $i = j$, and (iii) $a_{ij} = 0$ otherwise. The Kirchhoff matrix tree theorem states that all cofactors of A are equal, and their common value is $t(G)$. The matrix tree theorem can be applied to any graph G to determine $t(G)$, but this requires evaluating a determinant of

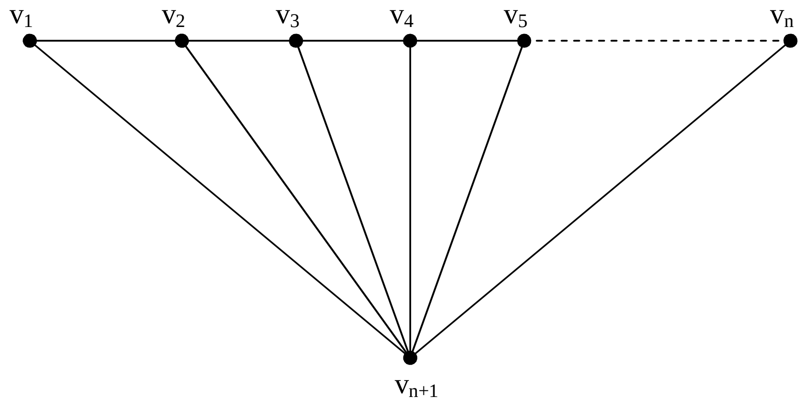

Figure 1: F_{n+1}

a corresponding characteristic matrix. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such results is due to Cayley who showed that complete graph on n vertices, K_n, has n^{n-2} spanning trees [3]. That is he showed

$$t(K_n) = n^{n-2} \quad \text{for } n \geq 2.$$

Another result is due to Sedlacek [6] who derived a formula for the wheel on $n+1$ vertices, W_{n+1}, which is formed from a cycle C_n on n vertices by adding a vertex adjacent to every vertex of C_n. In particular, he showed that

$$t(W_{n+1}) = \Big(\frac{3+\sqrt{5}}{2}\Big)^n + \Big(\frac{3-\sqrt{5}}{2}\Big)^n - 2 \quad \text{for } n \geq 3.$$

Sedlacek also later derived a formula for the number of spanning trees in a Möbius ladder [7]. The Möbius ladder, M_n, is formed from cycle C_{2n} on $2n$ vertices labeled $v_1, v_2, ..., v_{2n}$ by adding edge $v_i v_{i+n}$ for every vertex v_i, where $i \leq n$. The number of spanning trees in M_n equals

$$t(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2] \quad \text{for } n \geq 2.$$

Another class of graphs for which an explicit formula has been derived is based on a prism [2]. Let the vertices of two disjoint and equal length cycles be labeled $v_1, v_2, ..., v_n$ in one cycle and $w_1, w_2, ..., w_n$ in the other. The prism, R_n, is defined as the graph obtained by adding to these two cycles all edges of the form $v_i w_i$. The number of spanning trees in R_n is given by the following formula.

$$t(R_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2] \quad \text{for } n \geq 3.$$

Baron et al. [1] derived the formula for the number of spanning trees in a square of cycle, C_n^2, which is expressed for $n \geq 5$ as follows.

$$t(C_n^2) = nF\_\{n\},$$

where $F\_\{n\}$ is the n'th Fibonacci number. Similar results can also be found in [8].

In this work we derive an even simpler formula, which expresses the number of spanning trees in n-fan directly in terms of Fibonacci numbers, i.e., $t(F_{n+1}) = F\_\{2n\}$. The method applied is to establish a recursion that is satisfied by a Kirchhoff cofactor of the fan. We subsequently will show that the stated formulas are the solutions to the recursion. Let x be the shift operator, $a_i = xa_{i-1}$ and $x_0 = 1$. For recursion $\lambda_k a_{i+k} + \lambda_{k-1} a_{i+k-1} + ... + \lambda_0 a_i = 0$ with constants $\lambda_0, \lambda_1, ..., \lambda_k$ we shall say that sequence $\{a_i\}$ satisfies

$$\sum_{j=0}^{k} \lambda_j x^j = 0.$$

2 Main Results

We first form the Kirchhoff characteristic matrix A_{n+1} corresponding to the labeling shown in Figure 1. Next we focus our attention on the principal submatrix A_n obtained by canceling its last row and column corresponding to vertex v_{n+1} (Figure 2). So, the number of spanning trees of n-fan equals $t(F_{n+1}) = det(A_n)$.

$$A_n = \begin{array}{c} v_1 \; v_2 \; \ldots\ldots\ldots\ldots \; v_n \\ \left(\begin{array}{cccccc} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & -1 & 3 & -1 \\ & & & & -1 & 2 \end{array} \right) \end{array} \begin{array}{c} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_n. \end{array}$$

Figure 2: The principal tri-diagonal submatrix of F_{n+1}

Before presenting the main results we need the following lemma.

Lemma 2.1 *Let a_n denote the determinant of A_n. Then the sequence $\{a_n\}$ satisfies*

$$x^2 - 3x + 1 = 0$$

where x is the shift operator $a_n = xa_{n-1}$.

Proof: Expanding A_n along the first row we obtain

$$a_n = 2b_{n-1} - b_{n-2}.$$

where $b_i = det(B_i)$, and B_i is defined as in Figure 3.

$$B_i = \begin{pmatrix} 3 & -1 & & & & & \\ -1 & 3 & -1 & & & & \\ & -1 & 3 & -1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & -1 & 3 & -1 \\ & & & & & -1 & 2 \end{pmatrix}_{i \times i}$$

Figure 3: Auxiliary tri-diagonal matrix B_i

Now we derive the recursion for b_n by expanding $det(B_n)$ along the first row. We obtain

$$b_n - 3b_{n-1} + b_{n-2} = 0,$$

which means that sequence $\{b_i\}$ satisfies

$$x^2 - 3x + 1 = 0.$$

So, it follows that sequence $\{a_n\}$ satisfies the above recursion. □

Theorem 2.2 *If $n \geq 2$ then*

$$t(F_{n+1}) = 2\frac{((3-\sqrt{5})/2)^{n+1} - ((3+\sqrt{5})/2)^{n-1}}{5-3\sqrt{5}}.$$

Proof: We define the corresponding function

$$g(n) = 2\frac{((3-\sqrt{5})/2)^{n+1} - ((3+\sqrt{5})/2)^{n-1}}{5-3\sqrt{5}}.$$

By direct calculations we obtain that $g(n)$ satisfies

$$x^2 - 3x + 1 = 0,$$

where x is the shift operator $g(n) = xg(n-1)$. However, by Lemma 2.1, a_n satisfies the same recursion. In addition, we numerically evaluate $a_2, a_3, g(2), g(3)$ and find that $a_2 = g(2) = 3$ and $a_3 = g(3) = 8$. So, $a_n = g(n)$ and our formula holds. □

Theorem 2.3 *Let F_{i} be i'th Fibonacci number and* $n \geq 2$. *Then*

$$t(F_{n+1}) = F\_\{2n\}.$$

Proof: Clearly, Fibonacci numbers satisfy recursion $F\_\{2i\} - 3F\_\{2i-2\} + F\_\{2i-4\} = 0$ for $i > 2$. So, if we define $f(i)$ as $F\_\{2i\}$ then sequence $\{f(i)\}$ satisfies

$$x^2 - 3x + 1 = 0,$$

where x is the shift operator $f(i) = xf(i-1)$. By Lemma 2.1, a_n satisfies the same recursion. In addition, $F\_\{4\} = f(2) = a_2 = 3$, and $F\_\{6\} = f(3) = a_3 = 8$. So, $f(n) = a_n$ and our result follows. □

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