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A Common Fixed Point Theorem and its Application in Dynamic Programming

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Abstract

A common fixed point theorem for certain contractive type mappings is presented in this paper. As an application, the existence and uniqueness of common solution for a system of functional equations arising in dynamic programming is given. The results presented in this paper generalize some known results in the literature.

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1 Introduction and Preliminaries

Let f, g and h be mappings from a metric space (X, d) into itself, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and

$$\Phi = \{W : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a continuous function such that } 0 < W(r) < r \text{ for all } r \in \mathbb{R}^+ \setminus \{0\}\}.$$

The existence of common fixed points and solutions for several classes of contractive type mappings and functional equations and system of functional equations arising in dynamic programming, respectively, have been studied by many investigators, for example, see [1-19] and the references therein.

Ray [18] studied the existence of common fixed point for the following contractive type mappings:

$$d(fx, gy) \le d(hx, hy) - W(d(hx, hy)), \quad \forall x, y \in X. \tag{1.1}$$

Liu [5] gave a sufficient condition which ensures the existence of common fixed point for the contractive type mappings:

$$\begin{aligned} d(fx, gy) \le \max\{&d(hx, hy), d(hx, fx), d(hy, gy)\} \\ &- W(\max\{d(hx, hy), d(hx, fx), d(hy, gy)\}), \quad \forall x, y \in X. \end{aligned} \tag{1.2}$$

As suggested in Bellman and Lee [1], the basic form of the functional equations in dynamic programming is as follows :

$$f(x) = \sup_{y \in S} H(x, y, f(T(x, y))), \quad \forall x \in D, \tag{1.3}$$

where x and y denote the state and decision vectors, respectively, T denotes the transformation of the process and $f(x)$ denotes the optimal return function with the initial state x. The authors [2-4, 6-17, 19] studied the existence or uniqueness of solutions, common solutions, coincidence solutions, nonpositive solutions and nonnegative solutions for several classes of functional equations and systems of functional equations arising in dynamic programming by using various fixed point theorems, common fixed point theorems and coincidence point theorems, respectively.

The purpose of this paper is to establish a unique common fixed point theorem for four self mappings f, g, h and t on X which satisfy the condition of the type

$$d(fx, gy) \tag{1.4}$$

$$\leq \max\Big\{d(fx,tx), d(gy,hy), d(hy,tx), \frac{1}{2}\big[d(fx,hy)+d(gy,tx)\big],$$
$$\frac{d(fx,tx)d(gy,hy)}{1+d(hy,tx)}, \frac{d(fx,hy)d(gy,tx)}{1+d(hy,tx)}, \frac{d(fx,hy)d(gy,tx)}{1+d(fx,gy)}\Big\}$$
$$-W\Big(\max\Big\{d(fx,tx), d(gy,hy), d(hy,tx), \frac{1}{2}\big[d(fx,hy)+d(gy,tx)\big],$$
$$\frac{d(fx,tx)d(gy,hy)}{1+d(hy,tx)}, \frac{d(fx,hy)d(gy,tx)}{1+d(hy,tx)}, \frac{d(fx,hy)d(gy,tx)}{1+d(fx,gy)}\Big\}\Big)$$

for all $x, y \in X$, where $W \in \Phi$. As an application, we prove the existence and uniqueness of common solutions for a class of system of functional equations arising in dynamic programming. The results presented in this paper extend and unify some results in [5] and [18].

2 A Common Fixed Point Theorem

Our main result is as follows.

Theorem 2.1. *Let (X,d) be a complete metric space, f, g, h and t be four continuous mappings from X into itself satisfying $ft = tf$, $gh = hg$, $f(X) \subseteq h(X)$ and $g(X) \subseteq t(X)$. If there exists $W \in \Phi$ satisfying* (1.4)*, then f, g, h and t have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq h(X)$ and $g(X) \subseteq t(X)$, it follows that there exist two sequences $\{y_n\}_{n\geq 1}$ and $\{x_n\}_{n\geq 0}$ such that $y_{2n+1} = fx_{2n} = hx_{2n+1}$ for $n \geq 0$ and $y_{2n} = gx_{2n-1} = tx_{2n}$ for $n \geq 1$. Define $d_n = d(y_n, y_{n+1})$ for $n \geq 1$. We first show that

$$d_{n+1} \leq d_n - W(d_n), \quad \forall n \geq 1. \tag{2.1}$$

Let $n \geq 1$. By (1.4) for $x = x_{2n}$ and $y = x_{2n+1}$, we have

$$d_{2n+1}$$
$$= d(fx_{2n}, gx_{2n+1})$$
$$\leq \max\Big\{d(fx_{2n},tx_{2n}), d(gx_{2n+1},hx_{2n+1}), d(hx_{2n+1},tx_{2n}),$$
$$\frac{1}{2}\big[d(fx_{2n},hx_{2n+1})+d(gx_{2n+1},tx_{2n})\big], \frac{d(fx_{2n},tx_{2n})d(gx_{2n+1},hx_{2n+1})}{1+d(hx_{2n+1},tx_{2n})},$$
$$\frac{d(fx_{2n},hx_{2n+1})d(gx_{2n+1},tx_{2n})}{1+d(hx_{2n+1},tx_{2n})}, \frac{d(fx_{2n},hx_{2n+1})d(gx_{2n+1},tx_{2n})}{1+d(fx_{2n},gx_{2n+1})}\Big\}$$
$$-W\Big(\max\Big\{d(fx_{2n},tx_{2n}), d(gx_{2n+1},hx_{2n+1}), d(hx_{2n+1},tx_{2n}),$$

$$\frac{1}{2}\big[d(fx_{2n}, hx_{2n+1}) + d(gx_{2n+1}, tx_{2n})\big], \frac{d(fx_{2n}, tx_{2n})d(gx_{2n+1}, hx_{2n+1})}{1 + d(hx_{2n+1}, tx_{2n})},$$

$$\frac{d(fx_{2n}, hx_{2n+1})d(gx_{2n+1}, tx_{2n})}{1 + d(hx_{2n+1}, tx_{2n})}, \frac{d(fx_{2n}, hx_{2n+1})d(gx_{2n+1}, tx_{2n})}{1 + d(fx_{2n}, gx_{2n+1})}\Big\}\Big)$$

$$\leq \max\Big\{d_{2n}, d_{2n+1}, d_{2n}, \frac{1}{2}d(y_{2n+2}, y_{2n}), \frac{d_{2n}d_{2n+1}}{1 + d_{2n}}, 0, 0\Big\}$$

$$-W\Big(\max\Big\{d_{2n}, d_{2n+1}, d_{2n}, \frac{1}{2}d(y_{2n+2}, y_{2n}), \frac{d_{2n}d_{2n+1}}{1 + d_{2n}}, 0, 0\Big\}\Big),$$

which implies that

$$d_{2n+1} \leq \max\{d_{2n}, d_{2n+1}\} - W(\max\{d_{2n}, d_{2n+1}\}). \tag{2.2}$$

Suppose that $d_{2n+1} > d_{2n}$ for some $n \geq 1$. It follows that $d_{2n+1} \leq d_{2n+1} - W(d_{2n+1}) < d_{2n+1}$, which is a contradiction. From (2.2) we infer that $d_{2n+1} \leq d_{2n} - W(d_{2n})$ for all $n \geq 1$. Hence $d_{2n+1} \leq d_{2n}$ for all $n \geq 1$. Similarly, we have $d_{2n} \leq d_{2n-1} - W(d_{2n-1})$ for $n \geq 1$. That is, (2.1) holds. Therefore the series of nonnegative terms $\sum_{n=1}^{\infty} W(d_n)$ is convergent. Hence

$$\lim_{n\to\infty} W(d_n) = 0.$$

Since $\{d_n\}_{n\geq 1}$ is a nonnegative decreasing sequence, it converges to some point p. By the continuity of W we have

$$W(p) = \lim_{n\to\infty} W(d_n) = 0,$$

which means that $p = 0$. Hence $\lim_{n\to\infty} d_n = 0$.

In order to show that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\geq 1}$ is not a Cauchy sequence. Thus there exists a positive number ϵ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \quad 2m(k) > 2n(k) > 2k.$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality, so that

$$d(y_{2m(k)-2}, y_{2n(k)}) \leq \epsilon, \quad d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \tag{2.3}$$

It follows that for each even integer $2k$,

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2n(k)}, d_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

Using (2.3) and the above inequality we deduce that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.4}$$

By the triangle inequality we infer that for each even integer $2k$

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2m(k)-1},$$

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2m(k)-1} + d_{2n(k)}$$

and

$$|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2m(k)}, y_{2n(k)})| \le d_{2n(k)}.$$

In view of (2.4) and the above inequalities we arrive at

$$\epsilon = \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1})$$

$$= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}).$$

By virtue of (1.4), we get that

$$\begin{aligned}
& d(y_{2n(k)}, y_{2m(k)}) \\
& \le d_{2n(k)} + d(fx_{2n(k)}, gx_{2m(k)-1}) \\
& \le d_{2n(k)} + \max\Big\{ d(fx_{2n(k)}, tx_{2n(k)}), d(gx_{2m(k)-1}, hx_{2m(k)-1}), \\
& d(hx_{2m(k)-1}, tx_{2n(k)}), \frac{1}{2}\big[d(fx_{2n(k)}, hx_{2m(k)-1}) + d(gx_{2m(k)-1}, tx_{2n(k)})\big], \\
& \frac{d(fx_{2n(k)}, tx_{2n(k)}) d(gx_{2m(k)-1}, hx_{2m(k)-1})}{1 + d(hx_{2m(k)-1}, tx_{2n(k)})}, \\
& \frac{d(fx_{2n(k)}, hx_{2m(k)-1}) d(gx_{2m(k)-1}, tx_{2n(k)})}{1 + d(hx_{2m(k)-1}, tx_{2n(k)})}, \\
& \frac{d(fx_{2n(k)}, hx_{2m(k)-1}) d(gx_{2m(k)-1}, tx_{2n(k)})}{1 + d(fx_{2n(k)}, gx_{2m(k)-1})}\Big\} \\
& -W\Big(\max\Big\{ d(fx_{2n(k)}, tx_{2n(k)}), d(gx_{2m(k)-1}, hx_{2m(k)-1}), \\
& d(hx_{2m(k)-1}, tx_{2n(k)}), \frac{1}{2}\big[d(fx_{2n(k)}, hx_{2m(k)-1}) + d(gx_{2m(k)-1}, tx_{2n(k)})\big], \\
& \frac{d(fx_{2n(k)}, tx_{2n(k)}) d(gx_{2m(k)-1}, hx_{2m(k)-1})}{1 + d(hx_{2m(k)-1}, tx_{2n(k)})}, \\
& \frac{d(fx_{2n(k)}, hx_{2m(k)-1}) d(gx_{2m(k)-1}, tx_{2n(k)})}{1 + d(hx_{2m(k)-1}, tx_{2n(k)})},
\end{aligned}$$

$$
\begin{aligned}
&\qquad \frac{d(fx_{2n(k)}, hx_{2m(k)-1})d(gx_{2m(k)-1}, tx_{2n(k)})}{1+d(fx_{2n(k)}, gx_{2m(k)-1})}\Big\}\Big) \\
&= \max\Big\{d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), d(y_{2m(k)-1}, y_{2n(k)}), \\
&\qquad \frac{1}{2}\big[d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)})\big], \\
&\qquad \frac{d(y_{2n(k)+1}, y_{2n(k)})d(y_{2m(k)}, y_{2m(k)-1})}{1+d(y_{2m(k)-1}, y_{2n(k)})}, \\
&\qquad \frac{d(y_{2n(k)+1}, y_{2m(k)-1})d(y_{2m(k)}, y_{2n(k)})}{1+d(y_{2m(k)-1}, y_{2n(k)})}, \\
&\qquad \frac{d(y_{2n(k)+1}, y_{2m(k)-1})d(y_{2m(k)}, y_{2n(k)})}{1+d(y_{2n(k)+1}, y_{2m(k)})}\Big\} \\
&\quad -W\Big(\max\Big\{d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), d(y_{2m(k)-1}, y_{2n(k)}), \\
&\qquad \frac{1}{2}\big[d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)})\big], \\
&\qquad \frac{d(y_{2n(k)+1}, y_{2n(k)})d(y_{2m(k)}, y_{2m(k)-1})}{1+d(y_{2m(k)-1}, y_{2n(k)})}, \\
&\qquad \frac{d(y_{2n(k)+1}, y_{2m(k)-1})d(y_{2m(k)}, y_{2n(k)})}{1+d(y_{2m(k)-1}, y_{2n(k)})}, \\
&\qquad \frac{d(y_{2n(k)+1}, y_{2m(k)-1})d(y_{2m(k)}, y_{2n(k)})}{1+d(y_{2n(k)+1}, y_{2m(k)})}\Big\}\Big).
\end{aligned}
$$

As $k \to \infty$, we infer that

$$
\begin{aligned}
\epsilon &\le \max\Big\{0,0,\epsilon,\epsilon,0,\frac{\epsilon^2}{1+\epsilon},\frac{\epsilon^2}{1+\epsilon}\Big\} - W\Big(\max\Big\{0,0,\epsilon,\epsilon,0,\frac{\epsilon^2}{1+\epsilon},\frac{\epsilon^2}{1+\epsilon}\Big\}\Big) \\
&= \epsilon - W(\epsilon),
\end{aligned}
$$

which implies that $W(\epsilon) \le 0$. This is a contradiction. Thus $\{y_n\}_{n\ge1}$ is a Cauchy sequence. Therefore $\{y_n\}_{n\ge1}$ converges to a point $z \in X$ by completeness of X. It follows that

$$
\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} hx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} tx_{2n} = z.
$$

By the continuity of h, f, t and g, and $ft = tf$, $hg = gh$, we conclude that for any $n \ge 0$

$$
\begin{aligned}
&d(tfx_{2n}, hgx_{2n+1}) \\
&= d(ftx_{2n}, ghx_{2n+1})
\end{aligned}
$$

$$\le \max\Big\{d(ftx_{2n}, ttx_{2n}), d(ghx_{2n+1}, hhx_{2n+1}), d(hhx_{2n+1}, ttx_{2n}),$$
$$\frac{1}{2}\big[d(ftx_{2n}, hhx_{2n+1}) + d(ghx_{2n+1}, ttx_{2n})\big],$$
$$\frac{d(ftx_{2n}, ttx_{2n})d(ghx_{2n+1}, hhx_{2n+1})}{1 + d(hhx_{2n+1}, ttx_{2n})},$$
$$\frac{d(ftx_{2n}, hhx_{2n+1})d(ghx_{2n+1}, ttx_{2n})}{1 + d(hhx_{2n+1}, ttx_{2n})},$$
$$\frac{d(ftx_{2n}, hhx_{2n+1})d(ghx_{2n+1}, ttx_{2n})}{1 + d(ftx_{2n}, ghx_{2n+1})}\Big\}$$
$$-W\Big(\max\Big\{d(ftx_{2n}, ttx_{2n}), d(ghx_{2n+1}, hhx_{2n+1}), d(hhx_{2n+1}, ttx_{2n}),$$
$$\frac{1}{2}\big[d(ftx_{2n}, hhx_{2n+1}) + d(ghx_{2n+1}, ttx_{2n})\big],$$
$$\frac{d(ftx_{2n}, ttx_{2n})d(ghx_{2n+1}, hhx_{2n+1})}{1 + d(hhx_{2n+1}, ttx_{2n})},$$
$$\frac{d(ftx_{2n}, hhx_{2n+1})d(ghx_{2n+1}, ttx_{2n})}{1 + d(hhx_{2n+1}, ttx_{2n})},$$
$$\frac{d(ftx_{2n}, hhx_{2n+1})d(ghx_{2n+1}, ttx_{2n})}{1 + d(ftx_{2n}, ghx_{2n+1})}\Big\}\Big).$$

As $n \to \infty$, we get that

$$d(tz, hz) \le \ d(tz, hz) - w\big(d(tz, hz)\big),$$

which gives that $tz = hz$. Note that

$$d(tfx_{2n}, hgx_{2n+1}) = d(ftx_{2n}, ghx_{2n+1}), \quad \forall n \ge 0.$$

As $n \to \infty$, we gain immediately that $d(fz, gz) = d(hz, tz)$. Hence $fz = gz$.

It follows from (1.4)

$$d(fx_{2n}, hgx_{2n+1})$$
$$= d(fx_{2n}, ghx_{2n+1})$$
$$\le \max\Big\{d(fx_{2n}, tx_{2n}), d(ghx_{2n+1}, hhx_{2n+1}), d(hhx_{2n+1}, tx_{2n}),$$
$$\frac{1}{2}\big[d(fx_{2n}, hhx_{2n+1}) + d(ghx_{2n+1}, tx_{2n})\big],$$
$$\frac{d(fx_{2n}, tx_{2n})d(ghx_{2n+1}, hhx_{2n+1})}{1 + d(hhx_{2n+1}, tx_{2n})},$$
$$\frac{d(fx_{2n}, hhx_{2n+1})d(ghx_{2n+1}, tx_{2n})}{1 + d(hhx_{2n+1}, tx_{2n})},$$

$$\frac{d(fx_{2n},hhx_{2n+1})d(ghx_{2n+1},tx_{2n})}{1+d(fx_{2n},ghx_{2n+1})}\Big\}$$

$$-W\Big(\max\Big\{d(fx_{2n},tx_{2n}),d(ghx_{2n+1},hhx_{2n+1}),d(hhx_{2n+1},tx_{2n}),$$

$$\frac{1}{2}\big[d(fx_{2n},hhx_{2n+1})+d(ghx_{2n+1},tx_{2n})\big],$$

$$\frac{d(fx_{2n},tx_{2n})d(ghx_{2n+1},hhx_{2n+1})}{1+d(hhx_{2n+1},tx_{2n})},$$

$$\frac{d(fx_{2n},hhx_{2n+1})d(ghx_{2n+1},tx_{2n})}{1+d(hhx_{2n+1},tx_{2n})},$$

$$\frac{d(fx_{2n},hhx_{2n+1})d(ghx_{2n+1},tx_{2n})}{1+d(fx_{2n},ghx_{2n+1})}\Big\}\Big),\quad \forall n\ge 0.$$

As $n\to\infty$, we get that

$$d(z,hz)\le \max\Big\{0,0,d(z,hz),d(z,hz),0,\frac{d^2(z,hz)}{1+d(z,hz)},\frac{d^2(z,hz)}{1+d(z,hz)}\Big\}$$

$$-W\Big(\max\Big\{0,0,d(z,hz),d(z,hz),0,\frac{d^2(z,hz)}{1+d(z,hz)},\frac{d^2(z,hz)}{1+d(z,hz)}\Big\}\Big)$$

$$=d(z,hz)-W(d(z,hz)),$$

which implies that $z=hz$.

Using (1.4), we infer that

$$d(ffx_{2n},gx_{2n+1})$$

$$\le \max\Big\{d(ffx_{2n},tfx_{2n}),d(gx_{2n+1},hx_{2n+1}),d(hx_{2n+1},tfx_{2n}),$$

$$\frac{1}{2}\big[d(ffx_{2n},hx_{2n+1})+d(gx_{2n+1},tfx_{2n})\big],$$

$$\frac{d(ffx_{2n},tfx_{2n})d(gx_{2n+1},hx_{2n+1})}{1+d(hx_{2n+1},tfx_{2n})},$$

$$\frac{d(ffx_{2n},hx_{2n+1})d(gx_{2n+1},tfx_{2n})}{1+d(hx_{2n+1},tfx_{2n})},$$

$$\frac{d(ffx_{2n},hx_{2n+1})d(gx_{2n+1},tfx_{2n})}{1+d(ffx_{2n},gx_{2n+1})}\Big\}$$

$$-W\Big(\max\Big\{d(ffx_{2n},tfx_{2n}),d(gx_{2n+1},hx_{2n+1}),d(hx_{2n+1},tfx_{2n}),$$

$$\frac{1}{2}\big[d(ffx_{2n},hx_{2n+1})+d(gx_{2n+1},tfx_{2n})\big],$$

$$\frac{d(ffx_{2n},tfx_{2n})d(gx_{2n+1},hx_{2n+1})}{1+d(hx_{2n+1},tfx_{2n})},$$
$$\frac{d(ffx_{2n},hx_{2n+1})d(gx_{2n+1},tfx_{2n})}{1+d(hx_{2n+1},tfx_{2n})},$$
$$\frac{d(ffx_{2n},hx_{2n+1})d(gx_{2n+1},tfx_{2n})}{1+d(ffx_{2n},gx_{2n+1})}\Big\}\Big), \quad \forall n\geq 0.$$

Letting $n\to\infty$ in the above inequality, we get that

$$\begin{aligned} d(fz,z) \leq \max\Big\{&0,0,d(fz,z),d(fz,z),0,\frac{d^2(fz,z)}{1+d(fz,z)},\frac{d^2(fz,z)}{1+d(fz,z)}\Big\} \\ &-W\Big(\max\Big\{0,0,d(fz,z),d(fz,z), \\ &0,\frac{d^2(fz,z)}{1+d(fz,z)},\frac{d^2(fz,z)}{1+d(fz,z)}\Big\}\Big) \\ &= d(fz,z)-W(d(fz,z)), \end{aligned}$$

which means that $z=fz$. It follows that $z=fz=gz=tz=hz$. That is z is a common fixed point of f,g,h and t. If u is another common fixed point of f,g,h and t in X, it follow from (1.4) that

$$d(z,u)=d(fz,gu)\leq\ d(z,u)-w\big(d(z,u)\big)<d(z,u),$$

which is a contradiction. This completes the proof. □

As consequences of Theorem 2.1, we have the following results.

Corollary 2.2. *Let (X,d) be a complete metric space. Let f,g,h and t be four continuous mappings from X into itself, $ft=tf$, $gh=hg$, $f(X)\subseteq h(X)$ and $g(X)\subseteq t(X)$. If there exists $W\in\Phi$ satisfying*

$$d(fx,gy)\leq d(hy,tx)-W\big(d(hy,tx)\big), \quad \forall x,y\in X,$$

then f, g, h and t have a unique common fixed point in X.

Remark 2.3. Corollary 2.2 generalizes two results in [18].

Corollary 2.4. *Let (X,d) be a complete metric space. Let f, g and h be three continuous mappings from X into itself, $fh=hf$, $gh=hg$ and $f(X)\bigcup g(X)\subseteq h(X)$. If there exists $W\in\Phi$ satisfying*

$$\begin{aligned} &d(fx,gy) \\ &\leq \max\Big\{d(hx,hy),d(fx,hx),d(gy,hy),\frac{1}{2}\big[d(fx,hy)+d(gy,hx)\big]\Big\} \\ &-W\Big(\max\Big\{d(hx,hy),d(fx,hx),d(hy,gy),\frac{1}{2}\big[d(fx,hy)+d(gy,hx)\big]\Big\}\Big) \end{aligned}$$

for all $x,y\in X$, then f, g and h have a unique common fixed point in X.

Remark 2.5. Corollary 2.4 is a generalization of the Theorem and Corollaries 1 and 2 in [5].

3 An Application

Let X and Y be Banach spaces, $S \subseteq X$ be the state space, $D \subseteq Y$ be the decision space and i_X be the identity mapping on X. $B(S)$ denotes the set of all bounded real-valued functions on S and $d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}$. It is clear that $(B(S), d)$ is a complete metric space.

By means of Theorem 2.1, in this section we study the existence and uniqueness of common solution of the following system of functional equations arising in dynamic programming:

$$f_i(x) = \sup_{y \in D} \big\{u(x,y) + H_i(x,y,f_i(T(x,y)))\big\}, \quad \forall x \in S, i \in \{1,2,3,4\}, \tag{3.1}$$

where $u : S \times D \to \mathbb{R}$, $T : S \times D \to S$ and $H_i : S \times D \times \mathbb{R} \to \mathbb{R}$ for $i \in \{1,2,3,4\}$.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(a1) *u and H_i are bounded for $i \in \{1,2,3,4\}$;*

(a2) *There exist $W \in \Phi$ and the mappings A_1, A_2, A_3 and A_4 defined by*

$$A_i g_i(x) = \sup_{y \in D} \big\{u(x,y) + H_i(x,y,g_i(T(x,y)))\big\}, \ \forall x \in S, g_i \in B(S), i \in \{1,2,3,4\};$$

satisfying

$$\begin{aligned}
&|H_1(x,y,g(t)) - H_2(x,y,h(t))| \\
&\le \max\Big\{d(A_1g,A_4g), d(A_2h,A_3h), d(A_3h,A_4g), \\
&\frac{1}{2}\big[d(A_1g,A_3h) + d(A_2h,A_4g)\big], \frac{d(A_1g,A_4g)d(A_2h,A_3h)}{1+d(A_3h,A_4g)}, \\
&\frac{d(A_1g,A_3h)d(A_2h,A_4g)}{1+d(A_3h,A_4g)}, \frac{d(A_1g,A_3h)d(A_2h,A_4g)}{1+d(A_1g,A_2h)}\Big\} \\
&-W\Big(\max\Big\{d(A_1g,A_4g), d(A_2h,A_3h), d(A_3h,A_4g), \\
&\frac{1}{2}\big[d(A_1g,A_3h) + d(A_2h,A_4g)\big], \frac{d(A_1g,A_4g)d(A_2h,A_3h)}{1+d(A_3h,A_4g)}, \\
&\frac{d(A_1g,A_3h)d(A_2h,A_4g)}{1+d(A_3h,A_4g)}, \frac{d(A_1g,A_3h)d(A_2h,A_4g)}{1+d(A_1g,A_2h)}\Big\}\Big)
\end{aligned}$$

for all $(x,y) \in S \times D$, $g,h \in B(S), t \in S$;

(a3) *$A_1(B(S)) \subseteq A_3(B(S))$, $A_2(B(S)) \subseteq A_4(B(S))$;*

(a4) *There exists some $A_i \in \{A_1,A_2,A_3,A_4\}$ such that for any sequence $\{h_n\}_{n\ge1} \subseteq B(S)$ and $h \in B(S)$,*

$$\lim_{n\to\infty} \sup_{x \in S} |h_n(x) - h(x)| = 0 \Rightarrow \lim_{n\to\infty} \sup_{x \in S} |A_ih_n(x) - A_ih(x)| = 0;$$

(a5) $A_1A_4 = A_4A_1,\ A_2A_3 = A_3A_2$.

Then the system of functional equations (3.1) *has a unique common solution in* $B(S)$.

Proof. It follows from (a1)-(a4) that A_1, A_2, A_3 and A_4 are continuous self mappings of $B(S)$. For any $g, h \in B(S)$, $x \in S$ and $\varepsilon > 0$, there exist $y, z \in D$ such that

$$A_1g(x) < u(x,y) + H_1(x,y,g(T(x,y))) + \varepsilon, \tag{3.2}$$

$$A_2h(x) < u(x,z) + H_2(x,z,h(T(x,z))) + \varepsilon. \tag{3.3}$$

Note that

$$A_1g(x) \geq u(x,z) + H_1(x,z,g(T(x,z))), \tag{3.4}$$

$$A_2h(x) \geq u(x,y) + H_2(x,y,h(T(x,y))). \tag{3.5}$$

It follows from (3.2), (3.5) and (a2) that

$$\begin{aligned} & A_1g(x) - A_2h(x) \qquad (3.6)\\ & < H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y))) + \varepsilon \\ & \leq \max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g), \\ & \frac{1}{2}\big[d(A_1g, A_3h) + d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_3h)}{1 + d(A_3h, A_4g)}, \\ & \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_1g, A_2h)}\Big\} \\ & -W\Big(\max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g), \\ & \frac{1}{2}\big[d(A_1g, A_3h) + d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_3h)}{1 + d(A_3h, A_4g)}, \\ & \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_1g, A_2h)}\Big\}\Big) + \varepsilon. \end{aligned}$$

In view of (3.3), (3.4) and (a2) that

$$\begin{aligned} & A_1g(x) - A_2h(x) \qquad (3.7)\\ & > H_1(x,z,g(T(x,z))) - H_2(x,z,h(T(x,z))) - \varepsilon \end{aligned}$$

$$\geq -\max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g),$$

$$\frac{1}{2}\big[d(A_1g, A_3h) + d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_3h)}{1 + d(A_3h, A_4g)},$$

$$\frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_1g, A_2h)}\Big\}$$

$$+W\Big(\max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g),$$

$$\frac{1}{2}\big[d(A_1g, A_3h) + d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_4h)}{1 + d(A_3h, A_4g)},$$

$$\frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_1g, A_2h)}\Big\}\Big) - \varepsilon$$

(3.6) and (3.7) ensure that

$$\begin{aligned} & d(A_1g, A_2h) \\ = & \sup_{x\in S} |A_1g(x) - A_2h(x)| \\ \leq & \max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g), \\ & \frac{1}{2}\big[d(A_1g, A_3h) + d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_3h)}{1 + d(A_3h, A_4g)}, \\ & \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_1g, A_2h)}\Big\} \\ & -W\Big(\max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g), \\ & \frac{1}{2}\big[d(A_1g, A_3h) + d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_3h)}{1 + d(A_3h, A_4g)}, \\ & \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1 + d(A_1g, A_2h)}\Big\}\Big) + \varepsilon. \end{aligned} \tag{3.8}$$

Letting $\varepsilon \to 0$ in (3.8), we gain that

$$\begin{aligned} & d(A_1g, A_2h) \\ = & \sup_{x\in S} |A_1g(x) - A_2h(x)| \\ \leq & \max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g), \\ & \frac{1}{2}\big[d(A_1g, A_3h) + d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_3h)}{1 + d(A_3h, A_4g)}, \end{aligned} \tag{3.9}$$

$$\frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1+d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1+d(A_1g, A_2h)}\Big\}$$

$$-W\Big(\max\Big\{d(A_1g, A_4g), d(A_2h, A_3h), d(A_3h, A_4g),$$

$$\frac{1}{2}\big[d(A_1g, A_3h)+d(A_2h, A_4g)\big], \frac{d(A_1g, A_4g)d(A_2h, A_3h)}{1+d(A_3h, A_4g)},$$

$$\frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1+d(A_3h, A_4g)}, \frac{d(A_1g, A_3h)d(A_2h, A_4g)}{1+d(A_1g, A_2h)}\Big\}\Big).$$

It follows from (a5) and (3.9) that Theorem 2.1 implies that A_1, A_2, A_3 and A_4 have a unique common fixed point $v \in B(S)$, that is, $v(x)$ is a unique common solution of the system of functional equations (3.1). This completes the proof. □

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