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Minimizing the Probability of Ruin under Interest Force

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Abstract

In this paper, we consider a classical risk process model and allow investment into a risk-free asset as well as proportional reinsurance. The optimal proportional reinsurance strategy is found to minimize the probability of ruin. It is treated under two cases. The first case is a trivial case and the corresponding the minimal probability of ruin and the optimal proportional reinsurance strategy are given directly. For the second case, firstly the existence of the solution to the Hamilton-Jacobi-Bellman equation is proved. Then the minimal probability of ruin and the optimal proportional reinsurance strategy are obtained by a new verification theorem.

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1 Introduction

In recent years, stochastic control theory has gained much interest in insurance literature. This is due to the fact that the insurance company can control the surplus process so that a certain objective function is minimized (maximized). The corresponding approach under the classical risk model is pioneered by

Hipp and Plum (2000) who applied the classical stochastic control method to reduce the optimization problem to a matter of solving a Hamilton-Jabcobi-Bellman (HJB) equation. They found the optimal investment strategy to minimize the probability of ruin. Since their pioneering work many attempts have been made to solve the problem of minimizing the probability of ruin in a framework that allows more controls. Schmidli (2001) obtained optimal proportional reinsurance in the classical risk model, while optimal levels of reinsurance and investment are found by Schmidli (2002). In addition, Hipp and Vogt (2003) studied the same problem under the control of excess-of-loss reinsurance.

In this paper, we study the classical risk model with the possibility of investing in a risk-free asset as well as purchasing the proportional reinsurance, and find the optimal proportional reinsurance strategy to minimize the probability of ruin. Differently from Schmidli (2001) and Schmidli (2002), the corresponding HJB equation do not always have a smooth solution. In order to tackle the difficulty, we consider the optimal problem under two cases. Firstly a trivial case is shown. In this case, the corresponding minimal probability of ruin and the optimal proportional reinsurance strategy are given directly. For the second case, we prove the existence of solution to the HJB equation. Since the traditional verification theorem is no longer valid for our problem, a new verification theorem is given. Moreover, the maximal probability of survival and the optimal proportional reinsurance are obtained.

In section 2, the model assumptions are formulated, and a trivial case is shown. In section 3, the corresponding HJB equation is given and the existence of its solution is proved. In section 4, the verification theorem is given.

2 The model

We model the surplus of an insurance company by a compound poisson process. Here the number of claims N_t in $(0, t]$ is a poisson process with intensity λ, and the claim sizes Y_i $(i = 1, 2 \cdots)$ are a sequence of positive idd random variables independent of N_t. Let $G(x) = P(Y_i \leq x)$, $E[Y_i] = \mu$ and Y is a generic random variable which has the same distribution as Y_i $(i = 1, 2 \cdots)$. We assume that $G(x)$ is absolutely continuous. Let T_i be the occurrence time of the i-th claim. The premium income in $(0, t]$ is $(1+\eta)\lambda\mu t$ with safety loading $\eta > 0$. In addition to the premium income, the company also invests all of the surplus into a risk-free asset whose price process S_t^0 satisfies

$$dS_t^0 = rS_t^0 dt, \qquad r > 0.$$

Then, without reinsurance, the dynamics for the surplus process U_t is given by

$$dU_t = rU_t dt + (1+\eta)\lambda\mu dt - d\sum_{i=1}^{N_t} Y_i, \tag{1}$$

with the initial reserve $U_0 = u$.

For the rest of this paper we work on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which the process $\{U_t\}$ is defined. The information at time t is given by the complete filtration $\{\mathcal{F}_t\}$ generated by $\{U_t\}$.

A strategy α is described by a stochastic processes $\{b_t; t \geq 0\}$, where b_t represents the retention level at time t for reinsurance, which means that the insurer pays $b_t Y$ of a claim occurring at time t and the reinsurer pays $(1-b_t)Y$. For this reinsurance, the premium rate $\lambda\mu(1+\theta)(1-b_t)$ has to be paid, where θ represents the safety loading of the reinsurance company. We consider non-cheap reinsurance, that is $\theta > \eta$. Thus, when applying the strategy α, the resulting surplus process U_t^α is dynamicly given by

$$dU_t^\alpha = rU_t^\alpha dt + (b_t(1+\theta) - (\theta-\eta))\lambda\mu dt - d\sum_{i=1}^{N_t} b_{T_i} Y_i. \tag{2}$$

The strategy α is said to be admissible if b_t is $\mathcal{F}_t$ -progressively measurable and satisfies $0 \leq b_t \leq 1$ a.s. for all t. We denote by Π the set of all admissible strategies.

With the admissible strategy α, the ruin time is defined by

$$\tau_\alpha = \inf\{t \geq 0 : U_t^\alpha < 0\}.$$

Then the probability of ruin can be written as

$$\psi_\alpha(u) = P(\tau_\alpha < \infty | U_0^\alpha = u)$$

with the probability of survival

$$\phi_\alpha(u) = P(\tau_\alpha = \infty | U_0^\alpha = u) = 1 - \psi_\alpha(u).$$

The aim is to minimize the probability of ruin which is the same as maximizing the probability of survival. We will compute the maximal probability of survival.

$$\phi(u) = \sup_{\alpha\in\Pi} \phi_\alpha(u) \tag{3}$$

and find an optimal proportional reinsurance strategy α^* such that

$$\phi(u) = \phi_{\alpha^*}(u).$$

Note that $\phi(u) = 0$ for $u < 0$. In this case, any admissible strategy can be regarded as an optimal proportional reinsurance strategy.

Before solving the problem we consider a trivial case, that is, $ru > \lambda\mu(\theta-\eta)$ for the initial reserve u. In view of (2), if the strategy $b_t \equiv 0$ for all t is applied, ruin will not take place for ever, i.e., $\phi_\alpha(u) = 1$. By (3), it follows that

$$\phi(u) = 1, \quad u \geq \lambda\mu(\theta - \eta)/r \tag{4}$$

and the strategy

$$b_t \equiv 0, \quad t \geq 0, \tag{5}$$

is optimal.

In the following sections we consider the maximal survival probability $\phi(u)$ on $[0, (\theta - \eta)\mu/r)$ and the corresponding optimal strategy.

3 The HJB equation and the existence of its solution

To solve the above optimization problem, the dynamic programming approach described in Fleming and Soner (1993) is used. From standard arguments, we know that if $\phi(u)$ is continuously differentiable, then $\phi(u)$ satisfies the following HJB equation:

$$\sup_{b\in[0,1]} (ru + (b(1+\theta) - (\theta - \eta))\lambda\mu)\phi'(u) - \lambda\phi(u) + \lambda E[\phi(u - bY)] = 0. \tag{6}$$

As follows, the existence of a solution to (6) on [0, *displaystyle*$\lambda\mu(\theta - \eta)/r)$ is proved. This will be done through a monotonicity argument.

Theorem 3.1 *There exists a strictly increasing solution $V(x)$ to the HJB equation (6) on the interval $[0,\ \lambda\mu(\theta-\eta)/r)$, which is continuous on $[0,\ \lambda\mu(\theta-\eta)/r]$ with $V(u) \to 1$ as $u \to \lambda\mu(\theta - \eta)/r$, continuously differentiable on $(0,\ \lambda\mu(\theta - \eta)/r)$.*

Proof Let $\underline{b}(u)$ be the value where the equality

$$ru + (b(1+\theta) - (\theta - \eta))\lambda\mu = 0$$

holds for $0 \leq u < \frac{(\theta-\eta)\lambda\mu}{r}$, then

$$\underline{b}(u) = \frac{(\theta - \eta)\lambda\mu - ru}{(1+\theta)\lambda\mu} > 0.$$

Since we are looking for a strictly increasing solution of equation (6), we can rewrite it as

$$\phi^{'}(u) = \inf_{b\in(\underline{b}(u),1]} \lambda \frac{\phi(u) - E\phi(u-bY)}{ru + (b(1+\theta) - (\theta-\eta))\lambda\mu}, \; 0 \le u < \frac{(\theta-\eta)\lambda\mu}{r}. \tag{7}$$

Define a sequence $V_n(u)$ via $V_0(u) = \phi_0(u)$, the probability of ruin without reinsurance for $n = 0$, and through the recursion

$$V^{'}_{n+1}(u) = \inf_{b\in(\underline{b}(u),1]} \lambda \frac{V_n(u) - EV_n(u-bY)}{ru + (b(1+\theta) - (\theta-\eta))\lambda\mu}. \tag{8}$$

For $n = 0$ we have

$$V^{'}_0(u) = \lambda \frac{V_0(u) - EV_0(u-Y)}{ru + (1+\eta)\lambda\mu} > 0, \tag{9}$$

[see Sundt and Teugels (1995)], and from (8) we get for $n = 1$

$$V^{'}_1(u) = \inf_{b\in(\underline{b}(u),1]} \lambda \frac{V_0(u) - EV_0(u-bY)}{ru + (b(1+\theta) - (\theta-\eta))\lambda\mu}. \tag{10}$$

It follows that $V^{'}_1(u) \le V^{'}_0(u)$ for all $u \ge 0$. Let $b_i(u)$ $(i = n, n+1)$ are the points where the minimum is taken for $V^{'}_i(u)$ $(i = n, n+1)$ respectively. Then the considerations above show that

$$\begin{aligned} V^{'}_{n+1}(u) - V^{'}_n(u) &= \frac{E\int_{u-b_{n+1}(u)Y}^{u} V^{'}_n(s)ds}{ru+(b_{n+1}(u)(1+\theta)-(\theta-\eta))\lambda\mu} - \frac{E\int_{u-b_n(u)Y}^{u} V^{'}_{n-1}(s)ds}{ru+(b_n(u)(1+\theta)-(\theta-\eta))\lambda\mu} \\ &\le \frac{E\int_{u-b_n(u)Y}^{u} V^{'}_n(s) - V^{'}_{n-1}(s)ds}{ru + (b_n(u)(1+\theta) - (\theta-\eta))\lambda\mu} \le 0, \end{aligned}$$

and by induction $V^{'}_n(u)$ is a decreasing sequence. In addition, note that the minimum in (10) is not attained at the point $\underline{b}(u) > 0$, so $V^{'}_1(u) > 0$. By recursion $V^{'}_n(u) > 0$ for all $n \ge 1$. Thus the sequence $V^{'}_n(u)$ converges to a function $g(u)$. And with

$$V(u) = 1 - \int_u^{\frac{(\theta-\eta)\lambda\mu}{r}} g(s)ds, \tag{11}$$

we have a nondecreasing continuous function $V(u)$ satisfying

$$g(u) = \inf_{b\in(\underline{b}(u),1]} \lambda \frac{V(u) - EV(u-bY)}{ru + (b(1+\theta) - (\theta-\eta))\lambda\mu} \ge 0. \tag{12}$$

What is left is a proof for continuity of $g(u)$ and $g(u) > 0$, then

$$V^{'}(u) = g(u)$$

is continuous and $V(u)$ satisfies the equation (6) on [0, $displaystyle\lambda\mu(\theta-\eta)/r)$. Now we show that $g(u)>0$ for all $u>0$. First there exists $\varepsilon>0$ such that $g(u)>0$ for $u<\varepsilon$. Indeed, since $V_1'(u)>0$, we can rewrite (10) as

$$\sup_{b\in(\underline{b}(u),1]}(ru+(b(1+\theta)-(\theta-\eta)\lambda\mu)V_1'(u)+\lambda[EV_0(u-bY)-V_0(u)]=0 \quad (13)$$

with $b_1(0)=1$, where $b_1(u)$ is the maximizing function of (13). In view of lemma 3 in Schmidli (2002), there exists $\varepsilon>0$ such that $b_1(u)=1$ for $u<\varepsilon$. Thus from (9)-(10) equation $V_1'(u)=V_0'(u)$ holds for $u<\varepsilon$. By recursion, $V_n'(u)=V_0'(u)>0$ for $u<\varepsilon$ and all $n>0$. Therefore $g(u)=\lim_{n\to\infty}V_n'(u)=V_0'(u)>0$ for $u<\varepsilon$.

It remains to show that $g(u)>0$ is still true for $u\geq\varepsilon$. Otherwise we can assume that

$$u_0=\inf(u:g(u)=0)<\infty,$$

obviously $u_0\geq\varepsilon>0$ holds. Choose a ε_0 such that $\varepsilon_0\leq\varepsilon$ and

$$P(Y>\frac{\varepsilon_0(1+\theta)}{(\theta-\eta)\lambda\mu-r(u_0+\varepsilon_0)})>0. \quad (14)$$

Then there exists $u_0\leq u<u_0+\varepsilon_0$ for which $g(u)=0$ or

$$\inf_{b\in(\underline{b}(u),1]}[V(u)-EV(u-bY)]=V(u)-EV(u-\underline{b}(u)Y)=0,$$

i.e.

$$V(u)-EV(u-\frac{(\theta-\eta)\lambda\mu-ru}{1+\theta}Y)=0. \quad (15)$$

In addition, (14) yields

$$\begin{aligned}P(\frac{(\theta-\eta)\lambda\mu-ru}{1+\theta}Y>\varepsilon_0) &= P(Y>\frac{\varepsilon_0(1+\theta)}{(\theta-\eta)\lambda\mu-ru})\\ &> P(Y>\frac{\varepsilon_0(1+\theta)}{(\theta-\eta)\lambda\mu-r(u_0+\varepsilon_0)})>0. \quad (16)\end{aligned}$$

So, (15) gives

$$V(u)-EV(u-\frac{(\theta-\eta)\lambda\mu-ru}{1+\theta}Y)I_{(\frac{(\theta-\eta)\lambda\mu-ru}{1+\theta}Y>\varepsilon_0)}=0,$$

which implies

$$V(u)-V(u-\varepsilon_0)=0.$$

We obtain

$$0 = \int_{u-\varepsilon_0}^{u} g(s)ds \geq \int_{u-\varepsilon_0}^{u_0} g(s)ds \geq 0,$$

which contradicts the choice of u_0.

On the other hand, combining $g(u) > 0$ with (12) yields the continuity of $g(u)$.

4 The verification theorem

As is shown in Section 2, for the initial reserve $u < 0$ and $u \geq \lambda\mu(\theta-\eta)/r$, the maximal probability of survival $\phi(u)$ and corresponding optimal proportional reinsurance strategy α^* are given by

$$\phi(u) = \begin{cases} 0, & u < 0, \\ 1, & u \geq \lambda\mu(\theta-\eta)/r, \end{cases}$$

and

$$\alpha^* = \begin{cases} \text{any admissible strategy}, & u < 0, \\ \{b_t^* = 0, t \geq 0\}, & u \geq \lambda\mu(\theta-\eta)/r, \end{cases}$$

The following verification theorem shows

$$\phi(u) = V(u) \text{ of } (11), \quad 0 \leq u < \lambda\mu(\theta-\eta)/r,$$

and the corresponding optimal proportional reinsurance strategy is derived (see 17). Differently from the traditional verification theorem, it only involves the solution of the HJB equation (6) on $[0, \lambda\mu(\theta-\eta)/r)$ while the traditional verification theorem involves the solution of the HJB equation (6) on $[0, \infty)$. Moreover, the optimal proportional reinsurance strategy is skilly constructed.

Theorem 4.1 *Let $V(u)$ be defined by (11) with $V(u) = 0$ for $u < 0$. Then the maximal probability of survival $\phi(u) = V(u)$ on $[0, \lambda\mu(\theta-\eta)/r)$ and the optimal proportional reinsurance strategy α^* is given by*

$$b_t^* = \begin{cases} b^*(U_{t-}^{\alpha^*}), & t < T_{\alpha^*}, \\ 0, & t \geq T_{\alpha^*}, \end{cases} \tag{17}$$

where $T_{\alpha^*} = \inf\{t \geq 0 : \ U_t^{\alpha^*} = (\theta-\eta)\lambda\mu/r\}$ and $b^*(u)$ is the point on which the supremum is taken to the HJB equation (6) on $[0, \ \lambda\mu(\theta-\eta)/r)$.

Proof For any admissible strategy $\alpha = \{b_t\}$, let $T_\alpha = \inf\{t \geq 0 : U_t^\alpha = (\theta-\eta)\lambda\mu/r\}$. Define the strategy $\alpha^{'}$

$$b_t^{'} = \begin{cases} b_t, & t < T_\alpha, \\ 0, & t \geq T_\alpha. \end{cases}$$

The analysis in section 2 implies $\phi_{\alpha'}(u) \geq \phi_\alpha(u)$. This shows that we can restrict the optimization problem to such strategy α that $b_t = 0$ for $t \geq T_\alpha$. Let $\Pi^{'}$ denotes the set of such strategies. Then, for the strategy $\alpha^{'} \in \Pi^{'}$, applying Ito's formula into $V(U^{\alpha^{'}}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}})$ results in

$$V(U^{\alpha^{'}}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}}) = V(u) + \int_0^{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}} (rU_s^{\alpha^{'}} + b_s^{'}(1+\theta) - (\theta-\eta)\lambda\mu)V^{'}(U_s^{\alpha^{'}})ds + \sum_{i=1}^{N_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}}} (V(U^{\alpha^{'}}_{T_i}) - V(U^{\alpha^{'}}_{T_i-})) \tag{18}$$

By Brémaud (1981, page 27 or page 235),

$$\sum_{i=1}^{N_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}}} (V(U^{\alpha^{'}}_{T_i}) - V(U^{\alpha^{'}}_{T_i-})) - \lambda\int_0^{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}} [EV(U_s^{\alpha^{'}} - b_s^{'}Y) - V(U_s^{\alpha^{'}})]ds$$

is a martingale with zero-expectation. (Similarly also see Schmidli (2002) Theorem 1). Taking expectations on both side of (18) yields

$$\begin{aligned} V(u) = EV(U^{\alpha^{'}}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}}) - E\int_0^{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}} [rU_s^{\alpha^{'}} + (b_s^{'}(1+\theta) - (\theta-\eta)\lambda\mu)V^{'}(U_s^{\alpha^{'}}) \\ + \int_0^{U_s^{\alpha^{'}}/b_s^{\alpha^{'}}} \lambda V(U_s^{\alpha^{'}} - b_s^{'}y)dG(y) - \lambda V(U_s^{\alpha^{'}})]ds \geq EV(U^{\alpha^{'}}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}}), \end{aligned} \tag{19}$$

where the last inequality follows from (6). Note that

$$P(T_{\alpha'} < \tau_{\alpha'} | \tau_{\alpha'} = \infty) = 1. \tag{20}$$

Indeed, if $T_{\alpha'} = \tau_{\alpha'} = \infty$, $0 < U_t^{\alpha^{'}} < \lambda\mu(\theta-\eta)/r$ for all $t \geq 0$. In addition, there is a positive (maybe small) probability that

$$\sum_{i=N_t}^{N_{t+1}} b^{'}_{T_i}Y_i > \lambda\mu(\theta-\eta)/r + (1+\theta)\lambda\mu.$$

Therefore

$$\sum_{i=N_t}^{N_{t+1}} b^{'}_{T_i}Y_i > \lambda\mu(\theta-\eta)/r + \int_t^{t+1} rU_s^{\alpha^{'}}ds + \int_t^{t+1} b_s^{'}(1+\theta)\lambda\mu dt - (\theta-\eta)\lambda\mu$$

holds with a positive probability, where we use the fact that $U_s^{\alpha'} < \lambda\mu(\theta-\eta)/r$ on $\{t < T_{\alpha'}\}$ and $b_s^{'} \leq 1$. By the law of large numbers, ruin occurs, i.e., $\tau_{\alpha'} < \infty$, which contradicts $\tau_{\alpha'} = \infty$ and (20) is proved. Thus

$$\begin{aligned} V(u) &\geq EV(U^{\alpha'}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}}) = EV(U^{\alpha'}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}})I_{(T_{\alpha'}<\tau_{\alpha'},\tau_{\alpha'}=\infty)} \\ &+ EV(U^{\alpha'}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}})I_{(T_{\alpha'}>\tau_{\alpha'},\tau_{\alpha'}<\infty)} + EV(U^{\alpha'}_{t\wedge T_{\alpha'}\wedge\tau_{\alpha'}})I_{(T_{\alpha'}\leq\tau_{\alpha'},\tau_{\alpha'}<\infty)}, \end{aligned}$$

where the choice of the strategy α' implies that the last term is equal to 0. By the bounded convergence theorem as $t \to \infty$,

$$V(u) \geq EV(U^{\alpha'}_{\tau_{\alpha'}}) + P(\tau_{\alpha'} = \infty) = \phi_{\alpha'}(u),$$

Since the distribution $G(x)$ of claims is absolutely continuous , $U^{\alpha'}_{\tau_{\alpha'}} < 0$ a.s.. Thus, by $V(u) = 0$ for $u < 0$, we get

$$V(u) \geq P(\tau_{\alpha'} = \infty) = \phi_{\alpha'}(u),$$

which implies $V(u) \geq \phi(u)$.

Redoing the calculation with the strategy given by (17) yields $V(u) = EV(U_{t\wedge\tau_{\alpha'_*}\wedge T_{\alpha^*}})$ and letting $t \to \infty$ gives $V(u) = \phi_{\alpha^*}(u)$. Thus $V(u) \leq \phi(u)$ which ends the proof.

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