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On a Coupled Linear System with Homogeneous Damping

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Abstract

We investigate the global existence of both strong and weak solutions for a linear coupled system with homogeneous feedback boundary conditions in bounded-open domain Ω in $\mathbb{R}^n$ with $n \in \mathbb{N}$. We also prove the exponential decay of total energy associated with weak solutions.

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1 Introduction

The linear coupled system studied in this paper is motivated by both the wave and the heat linear equations. We will study the existence, uniqueness and exponential decay of solutions for the mixed problem

$$u''(x,t) + \mathcal{A}u(x,t) + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i}(x,t) = 0 \ \text{ in } \ \Omega\times]0,\infty[, \qquad (1.1)$$

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$$\theta'(x,t) + \mathcal{A}\theta(x,t) + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i}(x,t) = 0 \ \text{ in } \ \Omega\times]0,\infty[, \tag{1.2}$$

$$u(x,t) = 0, \quad \theta(x,t) = 0 \ \text{ on } \ \Gamma_0\times]0,\infty[, \tag{1.3}$$

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}}(x,t) + \alpha(x)u'(x,t) = 0 \ \text{ on} \Gamma_1\times]0,\infty[, \tag{1.4}$$

$$\frac{\partial \theta}{\partial \nu_{\mathcal{A}}}(x,t) + \beta\theta(x,t) = 0 \ \text{ on } \ \Gamma_1\times]0,\infty[, \tag{1.5}$$

$$u(x,0) = u^0(x), \ u'(x,0) = u^1(x), \ \theta(x,0) = \theta^0(x) \ \text{ in } \ \Omega, \tag{1.6}$$

where Ω is a bounded-open set in $\mathbb{R}^n$ with $n \in \mathbb{N}$. The boundary Γ of Ω is a C^2-set, and there exists a partition $\{\Gamma_0, \Gamma_1\}$ of Γ where Γ_0 and Γ_1 have positive measures and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is an empty set. The operator $\mathcal{A}$ and the co-normal derivative $\frac{\partial}{\partial \nu_{\mathcal{A}}}$ are given by

$$\mathcal{A}v(x,t) = -\sum_{i,\,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial v(x,t)}{\partial x_j}\Big) \quad \text{and} \quad \frac{\partial v}{\partial \nu_{\mathcal{A}}} = \sum_{i,\,j=1}^{n} a_{ij}(x,t)\frac{\partial v(x,t)}{\partial x_j}\nu_i.$$

We assume the following hypotheses on the functions of the system (1.1)-(1.6):

$$\alpha \in W^{1,\infty}(\Gamma_1), \ \ \alpha(x) \geq \alpha_0 > 0 \ \text{ and } \ \alpha_0\beta \geq \frac{n}{4} \ \text{ for } \ n \in \mathbb{N}. \tag{1.7}$$

$$\sum_{i,\,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \gamma\Big(|\xi_1|^2 + \ldots + |\xi_n|^2\Big) \text{ for all } \ \xi = (\xi_1,...,\xi_n) \in \Omega \ \text{ with } \ \gamma \geq 0. \tag{1.8}$$

$$a_{ij} = a_{ji} \ \text{ with } a_{ij} \in W^{1,\infty}_{\text{loc}}(0,\infty; C^1(\overline{\Omega})) \cap W^{2,\infty}_{\text{loc}}(0,\infty; L^2(\Omega)). \tag{1.9}$$

The subset V of $H^1(\Omega)$ is defined by $V = \{v \in H^1(\Omega); v = 0 \ \text{ on } \ \Gamma_0\}$ which is a Hilbert space with inner product and norm of $H^1(\Omega)$. In $V \times V$ we define the bilinear form

$$a(t,v,w) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t)\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_i}dx.$$

From (1.8) and (1.9) follows $a(t,v,w)$ is defined positive, symmetrical and continuous over $V \times V$. To get global solutions of the system (1.1)-(1.6) is also considered the additional hypotheses

$$\sum_{j=1}^{n} \frac{\partial}{\partial t}a_{jj}(x,t)\left(\frac{\partial v}{\partial x_j}\right)^2 \leq 0 \ \text{ for all } \ v \in V. \tag{1.10}$$

The constant γ defined in the hypothesis (1.8) satisfies

$$\gamma > \frac{nk_2}{2} \quad \text{where} \quad k_2 = \operatorname*{ess\ sup}_{\Omega\times]0,\infty[} |a'_{ij}(x,t)| \quad \text{and} \quad n \in \mathbb{N}. \tag{1.11}$$

We will use the Galerkin's method to show the existence of solutions for the system
(1.1)-(1.6). However, due to homogeneous boundary condition of feedback type (1.4) and (1.5), it is necessary the construction of a special basis to apply the Galerkin's method. In Medeiros-Milla Miranda [6] was constructed a special basis which allows authors to study some properties associated with wave equation with homogeneous boundary conditions via Galerkin method. To prove the global existence solutions for (1.1)-(1.6) we will make use of the ideas of Medeiros-Milla Miranda's paper. The asymptotic behavior of energy associated with weak solution of the system (1.1)-(1.6) will be determined assuming that $a_{ij}(x,t)$ is the real function $a(x,t)$ satisfying the properties fixed in (4.1). To obtain the exponential decay we construct a Liapunov operator and utilize the technics introduced by Haraux-Zuazua [1].

The one-dimensional thermoelastic system associated with (1.1)-(1.2), i.e., when $n = 1$ and $a_{ij}(x,t) \equiv 1$ with Dirichlet boundary conditions, has been studied for several authors. Questions about existence, uniqueness, stabilization asymptotic, and exact controllability of solutions has been answered. See, for instance, Hansen [7], Henry at al [2], Kim [3] among them.

2 Strong Solutions

The aim here is to prove the existence and uniqueness of solution for (1.1)-(1.6) when u^0, u^1 and θ^0 are smooth. Thus, we have the strong solution result

Theorem 2.1 *Assuming the hypotheses (1.7)-(1.11) and the initial data and boundary conditions satisfying*

$$u^0 \in V \cap H^2(\Omega), \quad u^1 \in V, \quad \theta^0 \in V \cap H^2(\Omega),$$
$$\frac{\partial u^0}{\partial \nu_{\mathcal{A}}} + \alpha(x) u^1 = 0 \;\; on \;\; \Gamma_1 \;\; and \;\; \frac{\partial \theta^0}{\partial \nu_{\mathcal{A}}} + \beta\theta^0 = 0 \;\; on \;\; \Gamma_1,$$

then there exists a unique pair of real functions $\{u, \theta\}$ solution of (1.1)-(1.6) such that

$$u \in L^\infty_{loc}(0,\infty; V \cap H^2(\Omega)), \quad u' \in L^\infty_{loc}(0,\infty; V), \quad u'' \in L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{2.1}$$
$$\theta \in L^\infty_{loc}(0,\infty; V \cap H^2(\Omega)) \cap L^2_{loc}(0,\infty; V), \quad \theta' \in L^\infty_{loc}(0,\infty; V), \tag{2.2}$$
$$u'' - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i} = 0 \;\; in \;\; L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{2.3}$$
$$\theta' - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial \theta}{\partial x_j}\Big) + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i} = 0 \;\; in \;\; L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{2.4}$$

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} + \alpha u' = 0 \quad in \quad L^{\infty}_{loc}(0,\infty; H^{1/2}(\Gamma_1)), \tag{2.5}$$

$$\frac{\partial \theta}{\partial \nu_{\mathcal{A}}} + \beta\theta = 0 \quad in \quad L^{\infty}_{loc}(0,\infty; H^{1/2}(\Gamma_1)) \cap L^{2}_{loc}(0,\infty; H^{3/2}(\Gamma_1)), \tag{2.6}$$

$$u(0) = u^0, \;\; u'(0) = u^1, \;\; \theta(0) = \theta^0. \tag{2.7}$$

Proof. In $V \cap H^2(\Omega)$ we define a special basis given by the sequences $(u^0_\ell)_{\ell\in\mathbb{N}}$, $(u^1_\ell)_{\ell\in\mathbb{N}}$ and $(\theta^0_\ell)_{\ell\in\mathbb{N}}$ such that

$$\begin{gathered} u^0_\ell \longrightarrow u^0 \text{ in } V \cap H^2(\Omega), \;\; u^1_\ell \longrightarrow u^1 \text{ in } V, \;\; \theta^0_\ell \longrightarrow \theta_0 \text{ in } V \cap H^2(\Omega) \text{ strongly}, \\ \frac{\partial u^0_\ell}{\partial \nu_{\mathcal{A}}} + \alpha u^1_\ell = 0 \text{ on } \Gamma_1 \text{ and } \frac{\partial \theta^0_\ell}{\partial \nu_{\mathcal{A}}} + \beta\theta^0_\ell = 0 \text{ on } \Gamma_1. \end{gathered} \tag{2.8}$$

For each $\ell \in \mathbb{N}$ fixed, and u^0_ℓ, u^1_ℓ and θ^0_ℓ linearly independent we define the vectors $\varpi^\ell_1 = u^0_\ell$, $\varpi^\ell_2 = u^1_\ell$ and $\varpi^\ell_3 = \theta^0_\ell$. Hence by orthornormalization process we construct a basis $\mathcal{B} = \{\varpi^\ell_1, \varpi^\ell_2, ..., \varpi^\ell_j, ...\}$ in $V \cap H^2(\Omega)$ satisfing (2.8). In these conditions, for each $m \in \mathbb{N}$ we consider the subspace $W^\ell_m = [\varpi^\ell_1, \varpi^\ell_2, ..., \varpi^\ell_m]$ generated by the m-first vectors of the basis $\mathcal{B}$. Thus, it is well known that the system of ordinary differential equations

$$(u''_{\ell m}(t), v) + a(t, u_{\ell m}(t), v) + \int_{\Gamma_1} \alpha(x) u'_{\ell m}(t) v d\Gamma + \sum_{i=1}^{n} \Big(\frac{\partial \theta_{\ell m}}{\partial x_i}(t), v\Big) = 0, \tag{2.9}$$

$$(\theta'_{\ell m}(t), w) + a(t, \theta_{\ell m}(t), w) + \beta \int_{\Gamma_1} \theta_{\ell m}(t) w \, d\Gamma + \sum_{i=1}^{n} \Big(\frac{\partial u'_{\ell m}}{\partial x_i}(t), w\Big) = 0, \tag{2.10}$$

$$u_{\ell m}(0) = u^0_\ell, \quad u'_{\ell m}(0) = u^1_\ell \quad \text{and} \quad \theta_{\ell m}(0) = \theta^0, \tag{2.11}$$

has a local solution $\{u_{\ell m}(t), \; \theta_{\ell m}(t)\}$ in $W^\ell_m \times W^\ell_m$ defined over $[0, t_m[$ for all $v, \; w \in W^\ell_m$. This pair of solutions can be extended over $[0, T]$ for any real number $T > 0$ thanks to estimates to proceed.

Substituting v by $u'_{\ell m}(t)$ into (2.9), w by $\theta_{\ell m}(t)$ into (2.10) and using the identities

$$\sum_{i=1}^{n} \Big(\frac{\partial u'_{\ell m}}{\partial x_i}(t), \theta_{\ell m}(t)\Big) = -\sum_{i=1}^{n} \Big(u'_{\ell m}(t), \frac{\partial \theta_{\ell m}}{\partial x_i}(t)\Big) + \sum_{i=1}^{n} \int_{\Gamma_1} u'_{\ell m}(t)\theta_{\ell m}(t)\nu_i d\Gamma,$$

$$a(t, u_{\ell m}(t), u'_{\ell m}(t)) = \frac{1}{2}\frac{d}{dt}\Big[a(t, u_{\ell m}(t), u_{\ell m}(t))\Big] - \frac{1}{2} a'(t, u_{\ell m}(t), u_{\ell m}(t)),$$

it follows from hypothesis (1.10) that

$$\frac{d}{dt}E_1(t) + \gamma\|\theta_{\ell m}(t)\|^2 + \Big(\alpha_0 - \frac{\epsilon}{2}\Big)\int_{\Gamma_1} [u'_{\ell m}(t)]^2 d\Gamma + \Big(\beta - \frac{n}{2\epsilon}\Big)\int_{\Gamma_1} [\theta_{\ell m}(t)]^2 d\,\Gamma \le 0, \tag{2.12}$$

where

$$E_1(t) = \frac{1}{2}\Big\{|u'_{\ell m}(t)|^2 + a(t, u_{\ell m}(t), u_{\ell m}(t)) + |\theta_{\ell m}(t)|^2\Big\}.$$

Now, differentiating the approximate equations (2.9) and (2.10) with respect to t, taking $v = u''_{\ell m}(t)$ and $w = \theta'_{\ell m}(t)$ it follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u''_{\ell m}|^2 + \frac{1}{2}\frac{d}{dt}|\theta'|^2 + a'(t, u_{\ell m}, u''_{\ell m}) + a(t, u'_{\ell m}, u''_{\ell m}) + a'(t, \theta_{\ell m}, \theta'_{\ell m}) + \\ a(t, \theta'_{\ell m}, \theta'_{\ell m}) + \int_{\Gamma_1} \alpha(x)[u''_{\ell m}]^2 d\Gamma + \beta \int_{\Gamma_1} [\theta'_{\ell m}]^2 d\Gamma + \\ \sum_{i=1}^{n} \Big(\frac{\partial \theta'_{\ell m}}{\partial x_i}, u''_{\ell m}\Big) + \sum_{i=1}^{n} \Big(\frac{\partial u''_{\ell m}}{\partial x_i}, \theta'_{\ell m}\Big) = 0. \end{aligned} \tag{2.13}$$

Next, we will analyze some terms in (2.13). Since

$$\sum_{i=1}^{n} \Big(\frac{\partial u''_{\ell m}}{\partial x_i}, \theta'_{\ell m}\Big) = -\sum_{\imath=1}^{n} \Big(u''_{\ell m}, \frac{\partial \theta'_{\ell m}}{\partial x_i}\Big) + \sum_{i=1}^{n} \int_{\Gamma_1} u''_{\ell m}\theta'_{\ell m}\nu_i d\Gamma, \quad \text{then for} \quad \epsilon > 0$$

$$\sum_{i=1}^{n} \Big(\frac{\partial u''_{\ell m}}{\partial x_i}, \theta'_{\ell m}\Big) \le -\sum_{i=1}^{n} \Big(u''_{\ell m}, \frac{\partial \theta'_{\ell m}}{\partial x_i}\Big) + \frac{\epsilon}{2}\int_{\Gamma_1} [u''_{\ell m}]^2 d\Gamma + \frac{n}{2\epsilon}\int_{\Gamma_1} [\theta'_{\ell m}]^2 d\Gamma. \tag{2.14}$$

Thanks to hypotheses (1.8) and (1.10) yields

$$\begin{aligned} a'(t, u_{\ell m}, u''_{\ell m}) \ge \frac{d}{dt}[a'(t, u_{\ell m}, u'_{\ell m})] - a''(t, u_{\ell m}, u'_{\ell m}), \\ a(t, u'_{\ell m}, u''_{\ell m}) = \frac{1}{2}\frac{d}{dt}[a(t, u'_{\ell m}, u'_{\ell m})] - \frac{1}{2}a'(t, u'_{\ell m}, u'_{\ell m}) \ge \frac{1}{2}\frac{d}{dt}[a(t, u'_{\ell m}, u'_{\ell m})], \\ a(t, \theta'_{\ell m}, \theta'_{\ell m}) \ge \gamma||\theta'_{\ell m}(t)||^2. \end{aligned} \tag{2.15}$$

Taking into account (2.14) and (2.15) into (2.13) yields

$$\begin{aligned} \frac{d}{dt}E_2(t) + \frac{d}{dt}[a'(t, u_{\ell m}, u'_{\ell m})] + \gamma||\theta'_{\ell m}||^2 + \Big(\alpha_0 - \frac{\epsilon}{2}\Big)\int_{\Gamma_1} [u''_{\ell m}]^2 d\Gamma + \\ \Big(\beta - \frac{n}{2\epsilon}\Big)\int_{\Gamma_1} [\theta'_{\ell m}]^2 d\Gamma \le |a''(t, u_{\ell m}, u'_{\ell m})| + |a'(t, \theta_{\ell m}, \theta'_{\ell m})|, \end{aligned} \tag{2.16}$$

where

$$E_2(t) = \frac{1}{2}\Big\{|u''_{\ell m}(t)|^2 + a(t, u'_{\ell m}(t), u'_{\ell m}(t)) + |\theta'_{\ell m}(t)|^2\Big\}.$$

Using the hypothesis (1.8) and usual inequalities into two terms of the right-hand side of (2.16) we can write

(a) $$|a''(t, u_{\ell m}, u'_{\ell m})| \le \frac{nk_1}{2}||u_{\ell m}||^2 + \frac{nk_1}{2}||u'_{\ell m}||^2 \le c_0 E_1(t) + c_1 E_2(t),$$

(b) $$|a'(t,\theta_{\ell m},\theta'_{\ell m})| \leq \frac{nk_2}{2}||\theta_{\ell m}||^2 + \frac{nk_2}{2}||\theta'_{\ell m}||^2,$$

where $k_1 = \underset{\Omega\times]0,\infty[}{\text{ess sup}} |a''_{ij}(x,t)|$ and $' = \dfrac{\partial}{\partial t}$. Taking into account **(a)** and **(b)** into (2.12) and (2.16) we have

$$\begin{aligned} &\frac{d}{dt}E(t) + \frac{d}{dt}[a'(t,u_{\ell m},u'_{\ell m})] + \Big(\gamma - \frac{nk_2}{2}\Big)\left\{\|\theta_{\ell m}(t)\|^2 + \|\theta'_{\ell m}(t)\|^2\right\} + \\ &\Big(\alpha_0 - \frac{\epsilon}{2}\Big)\int_{\Gamma_1}\left\{[u'_{\ell m}(t)]^2 + [u''_{\ell m}(t)]^2\right\}d\Gamma + \\ &\Big(\beta - \frac{n}{2\epsilon}\Big)\int_{\Gamma_1}\left\{[\theta_{\ell m}(t)]^2 + [\theta'_{\ell m}(t)]^2\right\}d\Gamma \leq cE(t), \end{aligned} \tag{2.17}$$

where $c = \max\{c_0,\ c_1\}$ and $E(t) = E_1(t) + E_2(t)$. Integrating (2.17) from 0 to t, with $0 \leq t \leq T$, and using the hypotheses (1.9), (1.10), and the hypotheses on initial conditions we have

$$\begin{aligned} &E(t) + \Big(\gamma - \frac{nk_2}{2}\Big)\int_0^t \Big\{\|\theta_{\ell m}(s)\|^2 + \|\theta'_{\ell m}(t)\|^2\Big\}ds + \\ &\Big(\alpha_0 - \frac{\epsilon}{2}\Big)\int_0^t\int_{\Gamma_1}\Big\{[u'_{\ell m}(s)]^2 + [u''_{\ell m}(s)]^2\Big\}d\Gamma ds + \\ &\Big(\beta - \frac{n}{2\epsilon}\Big)\int_0^t\int_{\Gamma_1}\Big\{[\theta_{\ell m}(s)]^2 + [\theta'_{\ell m}(s)]^2\Big\}d\Gamma ds \leq c_2 + E(0) + c\int_0^t E(s)ds, \end{aligned} \tag{2.18}$$

where c_2 depends on only of u^0 and u^1. To prove that $E(0)$ is bounded in V it is sufficient to show that $E_2(0)$ is bounded in V. It is equivalent to prove that $u''_{\ell m}(0)$ and $\theta'_{\ell m}(0)$ are bounded in $L^2(\Omega)$. In this point it will be clear the importance of the special base constructed previously. In fact, taking $t = 0$, $v = u''_{\ell m}(0)$ and $w = \theta'_{\ell m}(0)$ in (2.9) and (2.10) yields

$$(u''_{\ell m}(0), u''_{\ell m}(0)) + a(0, u^0_\ell, u''_{\ell m}(0)) + \int_{\Gamma_1}\alpha(x)u^1_\ell u''_{\ell m}(0)d\Gamma + \sum_{i=1}^n\Big(\frac{\partial\theta^0_\ell}{\partial x_i}, u''_{\ell m}(0)\Big) = 0, \tag{2.19}$$

$$(\theta'_{\ell m}(0), \theta'_{\ell m}(0)) + a(0, \theta^0_\ell, \theta'_{\ell m}(0)) + \beta\int_{\Gamma_1}\theta^0_\ell\theta'_{\ell m}(0)\,d\Gamma + \sum_{i=1}^n\Big(\frac{\partial u^1_\ell}{\partial x_i}, \theta'_{\ell m}(0)\Big) = 0. \tag{2.20}$$

From Green's formula it implies

$$\begin{aligned} a(0,u^0_\ell,u''_{\ell m}(0)) &= \sum_{i,j=1}^n\int_\Omega a_{ij}(x,0)\frac{\partial u^0_\ell}{\partial x_j}\frac{\partial u''_{\ell m}(0)}{\partial x_i}dx = \\ -\sum_{i,j=1}^n\int_\Omega\frac{\partial}{\partial x_i}\Big[a_{ij}(x,0)\frac{\partial u^0_\ell}{\partial x_j}\Big]u''_{\ell m}(0)dx &+ \sum_{i,j=1}^n\int_{\Gamma_1}a_{ij}(x,0)\frac{\partial u^0_\ell}{\partial x_j}\nu_i u''_{\ell m}(0)dx. \end{aligned}$$

From (2.9) we have $\sum_{i,j=1}^{n} a_{ij}\,(x,0)\,\frac{\partial u_\ell^0}{\partial x_j}\nu_i = -\alpha u_\ell^1 \quad on \quad \Gamma_1,$ thus

$$a(0,u_\ell^0,u_{\ell m}''(0)) = -\sum_{i,j=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}\Big[a_{ij}(x,0)\frac{\partial u_\ell^0}{\partial x_j}\Big]u_{\ell m}''(0)dx - \int_{\Gamma_1}\alpha(x)u^1 u_{\ell m}^{''}(0)d\,\Gamma. \quad (2.21)$$

$$a(0,\theta_\ell^0,\theta_{\ell m}^{'}(0)) = -\sum_{i,j=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}\Big[a_{ij}\,(x,0)\,\frac{\partial \theta_\ell^0}{\partial x_j}\Big]\theta_{\ell m}^{'}(0)dx - \beta\int_{\Gamma_1}\theta^0\theta_{\ell m}^{'}(0)d\,\Gamma. \quad (2.22)$$

Taking into account (2.22) and (2.22) into (2.19) and (2.20) respectively, yields

$$|u_{\ell m}^{''}(0)|^2 - \sum_{i,j=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}\Big[a_{ij}\,(x,0)\,\frac{\partial u_\ell^0}{\partial x_j}\Big]u_{\ell m}^{''}(0)dx + \sum_{i=1}^{n}\int_\Omega \frac{\partial \theta_\ell^0}{\partial x_i}u_{\ell m}^{''}(0)dx = 0,$$

$$|\theta_{\ell m}^{''}(0)|^2 - \sum_{i,j=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}\Big[a_{ij}(x,0)\frac{\partial \theta_\ell^0}{\partial x_j}\Big]\theta_{\ell m}^{'}(0)dx + \sum_{i=1}^{n}\int_\Omega \frac{\partial u_\ell^1}{\partial x_i}\theta_{\ell m}^{'}\,(0)\,dx = 0.$$

Hence, from convergence (2.8) and hypothesis (1.9) we get $u_{\ell m}''(0)$ and $\theta_{\ell m}''(0)$ are bounded in $L^2\,(\Omega)$. Therefore,

$$E_2(0) = \frac{1}{2}\Big\{|u_{\ell m}^{''}(0)|^2 + a\left(0,u_\ell^1,u_\ell^1\right) + |\theta_{\ell m}^{''}(0)|^2\Big\} \le C,$$

independent of ℓ and m. Consequently, from (2.18) and Gronwall's inequality

$$\begin{aligned} E(t) + \Big(\gamma - \frac{nk_2}{2}\Big)\int_0^t \Big\{\|\theta_{\ell m}(s)\|^2 + \|\theta_{\ell m}'(t)\|^2\Big\}ds + \\ \Big(\alpha_0 - \frac{\epsilon}{2}\Big)\int_0^t\int_{\Gamma_1}\Big\{[u_{\ell m}'(s)]^2 + [u_{\ell m}''(s)]^2\Big\}d\Gamma ds + \\ \Big(\beta - \frac{n}{2\epsilon}\Big)\int_0^t\int_{\Gamma_1}\Big\{[\theta_{\ell m}(s)]^2 + [\theta_{\ell m}'(s)]^2\Big\}d\Gamma ds \le C, \end{aligned} \quad (2.23)$$

for all t in $[0,T]$ and the constant C depends only the hypothesis (2.8). Note that all the constants on the left-hand side of (2.23) are positive, thanks to hypotheses (1.7) and (1.11). For $\ell \in \mathbb{N}$ fixed, the estimate (2.23) permit us by induction and diagonal process to obtain subsequences $(u_{\ell m_n})_{n\in\mathbb{N}}$ of $(u_{\ell m})_{m\in\mathbb{N}}$ and $(\theta_{\ell m_n})_{n\in\mathbb{N}}$ of $(\theta_{\ell m})_{m\in\mathbb{N}}$ which also will be denoted by $(u_{\ell m})_{m\in\mathbb{N}}$ and $(\theta_{\ell m})_{m\in\mathbb{N}}$ respectively. Besides, we also will get functions $u_\ell : \Omega\times]0,\infty[\longrightarrow \mathbb{R}$ and $\theta_\ell : \Omega\times]0,\infty[\longrightarrow \mathbb{R}$ as a consequence of the convergence of those subsequences. In these conditions, we multiply both sides of the approximate equations (2.9) and (2.10) by $\psi \in \mathcal{D}(0,\infty)$ and integrate in relation to t for all $v,w \in W_m^\ell$ which imply

$$\begin{aligned} \int_0^\infty (u_\ell''(t),v)\psi(t)\,dt + \int_0^\infty \mu(t)((u_\ell(t),v))\psi(t)\,dt + \\ \int_0^\infty\int_{\Gamma_1}\alpha(x)u_\ell'(t)\,v\,\psi(t)\,d\Gamma\,dt + \sum_{i=1}^{n}\int_0^\infty \Big(\frac{\partial \theta_\ell}{\partial x_i}(t),v\Big)\psi(t)\,dt = 0, \end{aligned} \quad (2.24)$$

$$\begin{aligned}&\int_0^\infty (\theta_\ell', w)\psi(t)\,dt + \int_0^\infty ((\theta_\ell(t), w))\psi(t)\,dt +\\ &\beta\int_0^\infty\int_{\Gamma_1}\theta_\ell(t)w\psi(t)d\Gamma + \sum_{i=1}^n\int_0^\infty \Big(\frac{\partial u_\ell'}{\partial x_i}(t), w\Big)\psi(t)\,dt = 0.\end{aligned} \tag{2.25}$$

As $\mathcal{B}=\left\{\varpi_1^\ell,\varpi_2^\ell,\varpi_3^\ell,\ldots\right\}$ is a Hilbertian basis of $V\cap H^2(\Omega)$, then by density we have (2.24) and (2.25) are still valid for all $v,\ w\in V\cap H^2(\Omega)$.

On the other hand, the estimate (2.23) also hold for all $\ell\in\mathbb{N}$. Thus, by the same process used in (2.24) and (2.25) we get two diagonal sequences (u_{ℓ_n}) and (θ_{ℓ_n}) which will still be denoted by (u_ℓ) and (θ_ℓ) and functions $u:\Omega\times]0,\infty[\longrightarrow\mathbb{R}$ and $\theta:\Omega\times]0,\infty[\longrightarrow\mathbb{R}$ such that

$$\begin{aligned}&\int_0^\infty (u''(t), v)\psi(t)dt + \int_0^\infty a(t,u(t),v)\psi(t)dt +\\ &\int_0^\infty\int_{\Gamma_1}\alpha(x)u'(t)v\psi(t)d\Gamma dt + \sum_{i=1}^n\int_0^\infty \Big(\frac{\partial \theta}{\partial x_i}(t), v\Big)\psi(t)\,dt = 0,\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\int_0^\infty (\theta'(t), w)\psi(t)dt + \int_0^\infty a(t,\theta(t),w)\psi(t)dt +\\ &\beta\int_0^\infty\int_{\Gamma_1}\theta(t)w\psi(t)d\Gamma\,dt + \sum_{i=1}^n\int_0^\infty \Big(\frac{\partial u'}{\partial x_i}(t), w\Big)\psi(t)\,dt = 0,\end{aligned} \tag{2.27}$$

for all ψ in $\mathcal{D}(0,\infty)$ and for all $v,w\in V$. As $\mathcal{D}(\Omega)$ is dense in V then (2.26) and (2.27) yield

$$u'' + \mathcal{A}u + \sum_{i=1}^n\frac{\partial\theta}{\partial x_i} = 0 \ \text{ in } \ L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)), \tag{2.28}$$

$$\theta' + \mathcal{A}\theta + \sum_{i=1}^n\frac{\partial u'}{\partial x_i} = 0 \ \text{ in } \ L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)). \tag{2.29}$$

Thus, the proof of (2.3) and (2.4) are completed ∎

From (2.23) and considering (2.26) and (2.27) we have $u,\ u'$ and θ belong to $L^\infty_{\text{loc}}(0,\infty;V)$. Hence,

$$\sum_{i=1}^n\frac{\partial\theta}{\partial x_i} \ \text{ and } \ \sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial\theta}{\partial x_j}\Big) \ \text{ belong to } \ L^2_{\text{loc}}(0,\infty;L^2(\Omega)), \tag{2.30}$$

$$\sum_{i=1}^n\frac{\partial u'}{\partial x_i} \text{ and } \sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) \text{ belong to } L^2_{\text{loc}}(0,\infty;L^2(\Omega)). \tag{2.31}$$

From (2.30), (2.31) and cf. [5] it follows $\frac{\partial u}{\partial\nu_{\mathcal{A}}}$ and $\frac{\partial\theta}{\partial\nu_{\mathcal{A}}}$ belong to $L^2_{\text{loc}}(0,\infty;H^{-1/2}(\Gamma_1))$. Multiplying (2.28) by $v\psi$ with $v\in V$ and $\psi\in\mathcal{D}(0,\infty)$

and (2.29) by $w\psi$ with $w \in V$ and $\psi \in \mathcal{D}(0,\infty)$ and using Green's formula it implies

$$\begin{aligned}&\int_0^\infty (u'',v)\psi dt + \int_0^\infty a(t,u,v)\psi dt - \int_0^\infty \Big\langle \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\nu_i, v\Big\rangle \psi dt +\\ &\sum_{i=1}^n \int_0^\infty \Big(\frac{\partial \theta}{\partial x_i}, v\Big)\psi dt = 0 \text{ for all } \psi \in \mathcal{D}(0,\infty) \text{ and for all } v \in V,\end{aligned} \tag{2.32}$$

$$\begin{aligned}&\int_0^\infty (\theta',w)\psi dt + \int_0^\infty a(t,\theta,w)\psi dt - \int_0^\infty \Big\langle \sum_{i,j=1}^n a_{ij}\frac{\partial \theta}{\partial x_j}\nu_i, w\Big\rangle \psi dt +\\ &\sum_{i=1}^n \int_0^\infty \Big(\frac{\partial u'}{\partial x_i}, w\Big)\psi dt = 0 \text{ for all } \psi \in \mathcal{D}(0,\infty) \text{ and for all } v \in V,\end{aligned} \tag{2.33}$$

where $<\ ,\ >$ is the duality paring of $H^{-1/2}(\Gamma_1)\times H^{1/2}(\Gamma_1)$. Comparing (2.26) and (2.27) with (2.32) and (2.33) respectively we obtain

$$\int_0^\infty \Big\langle \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\nu_i + \alpha(x)u',\ v\Big\rangle \psi dt = 0 \text{ for all } \psi \in \mathcal{D}(0,\infty) \text{ and for all } v \in V,$$

$$\int_0^\infty \Big\langle \sum_{i,j=1}^n a_{ij}\frac{\partial \theta}{\partial x_j}\nu_i + \beta\theta,\ w\Big\rangle \psi dt = 0 \text{ for all } \psi \in \mathcal{D}(0,\infty) \text{ and for all } v \in V.$$

Consequently,

$$\begin{aligned}&\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\nu_i + \alpha(x)u' = 0 \text{ in } L^\infty_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1)),\\ &\sum_{i,j=1}^n a_{ij}\frac{\partial \theta}{\partial x_j}\nu_i + \beta\theta = 0 \text{ in } L^\infty_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1)).\end{aligned} \tag{2.34}$$

Hence, we conclude (2.5) and (2.6) ▪

Now, we shall prove that $u \in L^\infty_{\text{loc}}(0,\infty;H^2(\Omega))$ and $\theta \in L^2_{\text{loc}}(0,\infty;H^2(\Omega))$. In fact, being $\{u,\theta\}$ solution of the boundary value problem

$$\begin{aligned}-\mathcal{A}u &= u'' + \sum_{i=1}^n \frac{\partial\theta}{\partial x_i} \quad \text{in} \quad \Omega\times[0,\infty[,\\ -\mathcal{A}\theta &= \theta' + \sum_{i=1}^n \frac{\partial u'}{\partial x_i} \quad \text{in} \quad \Omega\times[0,\infty[,\\ u &= \theta = 0 \quad \text{on} \quad \Gamma_0\times[0,\infty[,\\ \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\nu_i &+ \alpha(x)\,u' = 0 \quad \text{on} \quad \Gamma_1\times[0,\infty[,\\ \sum_{i,j=1}^n a_{ij}\frac{\partial \theta}{\partial x_j}\nu_i &+ \beta\theta = 0 \quad \text{on} \quad \Gamma_1\times[0,\infty[,\end{aligned}$$

we have from convergence of subsequence above, that

$$u'' + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i} \text{ belongs to } L^2_{\text{loc}}\left(0,\infty;L^2(\Omega)\right),$$
$$\theta' + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i} \text{ belongs to } L^2_{\text{loc}}\left(0,\infty;L^2(\Omega)\right).$$

Besides, (2.34) it implies that $\Upsilon_0(\alpha(x)\,u')$ and $\beta\Upsilon_0\theta$ belong to $L^\infty_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1))$, where Υ_0 is the trace operator of order zero. Thus, from results on elliptic regularity, it follows that u and θ belong to $L^\infty_{\text{loc}}(0,\infty;H^2(\Omega))$ ▪

Uniqueness of the solution is showed by standard energy method ▪

Other conclusions of Theorem 2.1 are verified in a standard way. Thus, the proof has been completed ■

Global solution of the system (1.1)-(1.6) is guaranteed by:

Corollary 2.1 *If in the hypotheses (1.9) we take* $a_{ij} \in W^{1,\infty}(0,\infty;C^1(\overline{\Omega})) \cap W^{2,\infty}(0,\infty;L^2(\Omega))$, *then there exists a unique pair of functions* $\{u,\theta\}$ *which satisfy the Theorem 2.1 for all* $t \in [0,\infty[$ ■

3 Weak Solutions

Our goal in this section is to obtain solutions for problem (1.1)–(1.6) with initial data

$$u^0 \in V, \quad u^1 \in L^2(\Omega) \quad \text{and} \quad \theta^0 \in V.$$

The corresponding solutions shall be called WEAK SOLUTION. To obtain that solution we approach u^0, θ^0 and u^1 by sequences of vectors in $V \cap H^2(\Omega)$ and V respectively, and apply the result of Theorem 2.1. The weak solution is guaranteed by the following theorem

Theorem 3.1 *Assuming the hypotheses (1.7)-(1.11) and* $u^0 \in V$, $u^1 \in L^2(\Omega)$ *and* $\theta^0 \in V$. *Then there exists a unique pair of real functions* $\{u,\theta\}$ *solution of (1.1)-(1.6) such that*

$$u \in C^0_b([0,\infty[;V) \cap C^1_b([0,\infty[;L^2(\Omega)), \tag{3.1}$$
$$\theta \in C^0_b([0,\infty[;L^2(\Omega)) \cap L^2_{loc}(0,\infty;V), \tag{3.2}$$
$$u'' - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i} = 0 \;\; \text{in } L^2_{loc}(0,\infty;V'), \tag{3.3}$$
$$\theta' - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial \theta}{\partial x_j}\Big) + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i} = 0, \;\; \text{in } L^2_{loc}(0,\infty;V') \tag{3.4}$$

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} + \alpha u' = 0 \quad in \quad L^2_{loc}(0,\infty; L^2(\Gamma_1)), \tag{3.5}$$

$$\frac{\partial \theta}{\partial \nu_{\mathcal{A}}} + \beta \theta = 0 \quad in \quad L^2_{loc}(0,\infty; L^2(\Gamma_1)), \tag{3.6}$$

$$u(0) = u^0, \quad u'(0) = u^1, \quad and \quad \theta(0) = \theta^0. \tag{3.7}$$

Proof. Let $(u^0_\ell)_{\ell\in\mathbb{N}}$ and $(\theta^0_\ell)_{\ell\in\mathbb{N}}$ be sequences of vectors in $V\cap H^2(\Omega)$ and $(u^1_\ell)_{\ell\in\mathbb{N}}$ sequences of vectors in V such that

$$u^0_\ell \longrightarrow u^0 \;\text{ in }\; V, \quad u^1_\ell \longrightarrow u^1 \;\text{ in }\; L^2(\Omega), \quad \theta^0_\ell \longrightarrow \theta^0 \;\text{ in }\; V,$$
$$\frac{\partial u^0_\ell}{\partial \nu_{\mathcal{A}}} + \alpha(x) u^1_\ell = 0 \;\text{ on }\; \Gamma_1 \quad\text{and}\quad \frac{\partial \theta^0_\ell}{\partial \nu_{\mathcal{A}}} + \beta\theta^0_\ell = 0 \;\text{ on }\; \Gamma_1.$$

In these conditions, there exists a sequence of strong solutions $\{u_\ell,\, \theta_\ell\}_{\ell\in\mathbb{N}}$ of system (1.1)-(1.6) defined by the precedent initial data, which satisfy the estimate (2.23). Thus, it implies in the existence of subsequence of $(u_\ell)_{\ell\in\mathbb{N}}$ and $(\theta_\ell)_{\ell\in\mathbb{N}}$ which will still be denoted by $(u_\ell)_{\ell\in\mathbb{N}}$ and $(\theta_\ell)_{\ell\in\mathbb{N}}$, and functions $u,\ \theta : \Omega\times[0,\infty[\to\mathbb{R}$ and $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2 : \Gamma_1\times[0,\infty[\to\mathbb{R}$ such that

$$u_\ell \rightharpoonup u \quad \text{weak star in} \quad L^\infty_{\text{loc}}(0,\infty; V), \tag{3.8}$$

$$u'_\ell \rightharpoonup u' \quad \text{weak star in} \quad L^\infty_{\text{loc}}(0,\infty; L^2(\Omega)), \tag{3.9}$$

$$\theta_\ell \rightharpoonup \theta \quad \text{weak in} \quad L^2_{\text{loc}}(0,\infty; V), \tag{3.10}$$

$$u'_\ell \rightharpoonup \varphi_1 \quad \text{weak in} \quad L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1)), \tag{3.11}$$

$$\sum_{i,j=1}^n a_{ij}\frac{\partial u_\ell}{\partial x_j}\nu_i \rightharpoonup \varphi_2 \quad \text{weak in} \quad L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1)), \tag{3.12}$$

$$\theta_\ell \rightharpoonup \psi_1 \quad \text{weak in} \quad L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1)), \tag{3.13}$$

$$\sum_{i,j=1}^n a_{ij}\frac{\partial \theta_\ell}{\partial x_i}\nu_i \rightharpoonup \psi_2 \quad \text{weak in} \quad L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1)). \tag{3.14}$$

It also has from (2.3) and (2.4) that

$$u''_\ell - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u_\ell}{\partial x_j}\Big) + \sum_{i=1}^n \frac{\partial \theta_\ell}{\partial x_i} = 0, \tag{3.15}$$

$$\theta'_\ell - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial \theta_\ell}{\partial x_j}\Big) + \sum_{i=1}^n \frac{\partial u'_\ell}{\partial x_i} = 0, \tag{3.16}$$

where both identities are in the sense of $L^\infty_{\text{loc}}(0,\infty; L^2(\Omega)$. Now, multiplying both side of (3.15) and (3.16) by $v\psi$ and $w\psi$ respectively, where $v, w \in V$ and $\psi \in \mathcal{D}(0,\infty)$, and integrating on $[0,\infty[$. After these, taking limit $\ell\to\infty$ and observing the convergence (3.8)-(3.14) it follows

$$-\int_0^\infty (u',v)\,\psi' dt + \int_0^\infty a\,(t,u,v)\,\psi dt + \int_0^\infty \int_{\Gamma_1} \alpha\,(x)\,\varphi_1 v\psi d\Gamma dt + \sum_{i=1}^n \int_0^\infty \Big(\frac{\partial\theta}{\partial x_i}, v\Big)\psi dt = 0, \tag{3.17}$$

$$-\int_0^\infty (\theta,v)\,\psi' dt + \int_0^\infty a\,(t,\theta,w)\,\psi dt + \beta\int_0^\infty \int_{\Gamma_1} \theta w\psi_1 d\Gamma dt + \sum_{i=1}^n \int_0^\infty \Big(\frac{\partial u'}{\partial x_i}, w\Big)\psi dt = 0. \tag{3.18}$$

The equations in (3.17) and (3.18) are still also valid with $v, w \in \mathcal{D}\,(\Omega)$. Thus, it follows

$$u'' - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}\,(x,t)\,\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^n \frac{\partial\theta}{\partial x_i} = 0, \tag{3.19}$$

$$\theta' - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}\,(x,t)\,\frac{\partial \theta}{\partial x_j}\Big) + \sum_{i=1}^n \frac{\partial u'}{\partial x_i} = 0, \tag{3.20}$$

where both identities are in the sense of $H^{-1}_{loc}(0,\infty;L^2(\Omega))$. Now, we will prove the identities

$$\varphi_2 = \sum_{i,j=1}^n \Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)\nu_i, \quad \psi_2 = \sum_{i,j=1}^n \Big(a_{ij}\frac{\partial \theta}{\partial x_j}\Big)\nu_i, \quad \varphi_1 = u' \quad \text{and} \quad \psi_1 = \theta.$$

From (2.5), (2.6) and (3.11)-(3.14) it follows

$$\varphi_2 + \alpha(x)\varphi_1 = 0 \ \text{ on } \ \Gamma_1 \times [0,\infty[\qquad \text{and} \qquad \psi_2 + \beta\psi_1 = 0 \ \text{ on } \ \Gamma_1 \times [0,\infty[. \tag{3.21}$$

Then it follows from (3.8)-(3.10) that

$$u_\ell \rightharpoonup u \quad \text{weak in} \quad L^2_{\text{loc}}\,(0,\infty;V)\,, \tag{3.22}$$

$$u'_\ell \rightharpoonup u' \quad \text{weak in} \quad L^2_{\text{loc}}\,\big(0,\infty;L^2\,(\Omega)\big)\,, \tag{3.23}$$

$$\theta_\ell \rightharpoonup \theta \quad \text{weak in} \quad L^2_{\text{loc}}\,(0,\infty;V)\,. \tag{3.24}$$

From (3.23) and (3.24) yields

$$u''_\ell \rightharpoonup u'' \quad \text{weak in} \quad H^{-1}_{\text{loc}}\,\big(0,\infty;L^2\,(\Omega)\big)\,, \tag{3.25}$$

$$\frac{\partial\theta_\ell}{\partial x_i} \rightharpoonup \frac{\partial\theta}{\partial x_i} \quad \text{weak in} \quad L^2_{\text{loc}}\,\big(0,\infty;L^2\,(\Omega)\big)\,. \tag{3.26}$$

From (3.21), (3.25) and (3.26) we get the weak convergence in $H^{-1}_{\text{loc}}(0,\infty;L^2(\Omega))$ of the term

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u_\ell}{\partial x_j}\Big) \rightharpoonup \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big). \tag{3.27}$$

Denoting by Υ_ι, $\iota = 0, 1$, the Trace operator and using (3.22) and (3.27) it implies (cf. Milla Miranda [5]) that $\Upsilon_1 u_\ell \rightharpoonup \Upsilon_1 u$ weak in $H^{-1}_{\text{loc}}(0,\infty; H^{-1/2}(\Gamma_1))$. It also, from (3.12) yields $\Upsilon_1 u_\ell \rightharpoonup \varphi_2$ weak in $L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1))$. Therefore,

$$\varphi_2 = \Upsilon_1 u = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \nu_i \quad \blacksquare \tag{3.28}$$

On the other hand, from (3.8) it results $u'_\ell \rightharpoonup u'$ weak in $H^{-1}_{\text{loc}}(0,\infty; H^{1/2}(\Gamma_1))$. Hence and from (3.11)

$$\varphi_1 = \Upsilon_0 u' \quad \blacksquare \tag{3.29}$$

From (3.22) and (3.24) it follows

$$\frac{\partial u'_\ell}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i}' \quad \text{weak in} \quad H^{-1}_{\text{loc}}(0,\infty; L^2(\Omega)), \tag{3.30}$$

$$\theta'_\ell \rightharpoonup \theta' \quad \text{weak in} \quad H^{-1}_{\text{loc}}(0,\infty; V). \tag{3.31}$$

Thus, from (3.20), (3.30) and (3.31) yields the weak convergence in $H^{-1}_{\text{loc}}(0,\infty; L^2(\Omega))$ of

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial \theta_\ell}{\partial x_j}\Big) \rightharpoonup \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial \theta}{\partial x_j}\Big). \tag{3.32}$$

The convergence (3.24) and (3.32) imply $\Upsilon_1\theta_\ell \rightharpoonup \Upsilon_1\theta$ weak in $H^{-1}_{\text{loc}}(0,\infty; H^{1/2}(\Gamma_1))$. It also follows from (3.14) that $\Upsilon_1\theta_\ell \rightharpoonup \psi_2$ weak in $L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1))$. Thus, we conclude

$$\psi_2 = \Upsilon_1 \theta = \sum_{i,j=1}^{n} a_{ij} \frac{\partial \theta}{\partial x_j} \nu_i \quad \blacksquare \tag{3.33}$$

On the other hand, from (3.26) it implies $\Upsilon_0\theta_\ell \rightharpoonup \Upsilon_0\theta$ weak in $L^2_{\text{loc}}(0,\infty; H^{1/2}(\Gamma_1))$. Hence and from (3.13) it follows

$$\psi_1 = \Upsilon_0 \theta \quad \blacksquare \tag{3.34}$$

From (3.21), (3.28) and (3.29) yields

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \nu_i + \alpha(x)\, u' = 0 \quad \text{in} \quad L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1)). \tag{3.35}$$

Analogously, from (3.15), (3.21), (3.33) and (3.34) yields

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial \theta}{\partial x_j} \nu_i + \beta\theta = 0 \quad \text{in} \quad L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1)). \tag{3.36}$$

Our next task is to prove the identities in (3.3) and (3.4) are satisfied in the sense of $L^2_{\text{loc}}(0,\infty;V')$. In fact, for all $v,\ w\in V$ it follows

$$\Big|\Big\langle -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big),v\Big\rangle\Big| \le |a(t,u,v)| + |\langle \Upsilon_1 u, v\rangle|_{H^{-1/2}(\Gamma_1)\times H^{1/2}(\Gamma_1)}\,,$$

where $\Upsilon_1 u = \displaystyle\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\nu_i$. Thus,

$$\Big|\Big\langle -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big),v\Big\rangle\Big| \le C\,(u)\,\|v\|_V \quad \text{for all} \quad v\in V. \tag{3.37}$$

$$\Big|\Big\langle -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial \theta}{\partial x_j}\Big),w\Big\rangle\Big| \le C\,(\theta)\,\|w\|_V \quad \text{for all} \quad w\in V, \tag{3.38}$$

where $C\,(u) = C\,\|u\|_V + C_0\Big|\dfrac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|_{H^{-1/2}(\Gamma_1)}$ and $C\,(\theta) = C\,\|\theta\|_V + C_0\Big|\dfrac{\partial \theta}{\partial \nu_{\mathcal{A}}}\Big|_{H^{-1/2}(\Gamma_1)}$.

From (3.37) and (3.38) we get for all $T>0$

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) \quad \text{and} \quad -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial \theta}{\partial x_j}\Big) \quad \text{belong to} \quad L^2(0,T;V'). \tag{3.39}$$

From (3.17), (3.18), (3.39), and Green's formula yields

$$\begin{aligned} &-\int_0^T (u',v)\,\psi' dt + \int_0^T \Big\langle -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big),v\Big\rangle\psi dt + \\ &\sum_{i=1}^{n}\int_0^T \Big(\frac{\partial \theta}{\partial x_i},v\Big)\psi dt = 0, \ \ \forall\ \psi\in\mathcal{D}\,(0,T) \ \text{ and } \ \forall\ v\in V, \end{aligned} \tag{3.40}$$

$$\begin{aligned} &-\int_0^T (\theta,w)\,\psi' dt + \int_0^T \Big\langle -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial \theta}{\partial x_j}\Big),w\Big\rangle\psi dt + \\ &\sum_{i=1}^{n}\int_0^T \Big(\frac{\partial u'}{\partial x_i},w\Big)\psi dt = 0, \ \ \forall\ \psi\in\mathcal{D}\,(0,T) \ \text{ and } \ \forall\ w\in V. \end{aligned} \tag{3.41}$$

Thus, from (3.39), (3.40), (3.41), and as $T>0$ is arbitrary we conclude (3.3) and (3.4) ▪

The regularities (3.1) and (3.2) are obtained in an usual way proving that $(u_\ell)_{\ell\in\mathbb{N}}$ and $(\theta_\ell)_{\ell\in\mathbb{N}}$ are Cauchy sequences ▪

The initial conditions $u(0)=u^0$, $u'(0)=u^1$, and $\theta(0)=\theta^0$ in Ω are obtained from regularities (3.1) and (3.2) ▪

Finally, the uniqueness of solutions is obtained using the method of regularization cf. Lions-Magenes [4] - pp 221, and also, Visik-Ladyzhenskaya [8] ▪

Thus, the proof of Theorem 3.1 is completed ■

4 Asymptotic Behavior

The aim of this section is to prove that the total energy associated with weak solutions of (1.1)-(1.6) has exponential decay as $t \longrightarrow +\infty$. To make this, we consider $\alpha(x) = m(x) \cdot \nu(x)$ and the following representation for Γ_0 and Γ_1

$$\Gamma_0 = \{x \in \Gamma;\ m(x) \cdot \nu(x) \leq 0\} \ \text{ and } \ \Gamma_1 = \{x \in \Gamma;\ \ m(x) \cdot \nu(x) > 0\},$$

where $m(x) = x - x^0$, for all $x \in \mathbb{R}^n$, $x^0 \in \mathbb{R}^n$ fixed, and $''\cdot''$ denotes the usual scalar product in $\mathbb{R}^n$. Herein, the operator $\mathcal{A}$ and the co-normal derivative of $\mathcal{A}$ are defined by

$$\mathcal{A}v(x,t) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(a(x,t)\frac{\partial v(x,t)}{\partial x_j}\Big) \quad \text{and} \quad \frac{\partial v}{\partial \nu_{\mathcal{A}}} = \sum_{j=1}^{n} a(x,t)\frac{\partial v(x,t)}{\partial x_j}\nu_j,$$

where the real function $a(x,t)$ satisfies

$$\begin{gathered} a(x,t) \in W^{1,\infty}_{\text{loc}}(0,\infty; C^1(\overline{\Omega})) \cap W^{2,\infty}_{\text{loc}}(0,\infty; L^2(\Omega)), \\ a(x,t) \geq a_0 > 0, \quad ||a||_\infty \leq \frac{\kappa\gamma}{R(x^0)}, \end{gathered} \tag{4.1}$$

where $||a||_\infty = \operatorname*{ess\ sup}_{\Omega\times]0,\infty[} \left|\frac{\partial}{\partial x_k} a(x,t)\right|$, $R(x^0) = \max_{x\in\overline{\Omega}} |m(x)|$ and κ is a real positive number.

The total energy $E(t)$ of the system (1.1)-(1.6) is given by

$$E(t) = \frac{1}{2}\Big\{|u'(t)|^2 + a(t,u(t),u(t)) + |\theta(t)|^2\Big\}$$

where

$$a(t,v,w) = \sum_{j=1}^{n} \int_\Omega a(x,t)\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_j}dx.$$

It is easy to see from (2.12) and (1.7) that the energy $E(t)$ is non increasing in view of

$$\frac{d}{dt}E(t) + \int_{\Gamma_1}\Big(\alpha_0 - \frac{\epsilon}{2}\Big)[u']^2 d\Gamma \leq -\gamma\|\theta\|^2. \tag{4.2}$$

In these conditions we have the following stability's result

Theorem 4.1 *If the real function $a(x,t)$ satisfies the hypothesis (4.1) then the energy $E(t)$ associated with weak solution of (1.1)-(1.6) guaranteed by Theorem 3.1 satisfies*

$$E(t) \leq \Lambda E(0)\exp(-\zeta\, t) \quad \textit{for all} \quad t \geq 0, \tag{4.3}$$

where Λ and ζ are real positive constants.

Proof. First we will get (4.3) to energy $E(t)$ given by strong solutions of (1.1)-(1.6). The stability's result for the energy associated with weak solution will be guaranteed by density properties. Therefore, let $\rho(t)$ be defined by

$$\rho(t) = 2(u'(t), m \cdot \nabla u(t)) + (n-1)(u'(t), u(t)). \tag{4.4}$$

Hence

$$\begin{aligned} \rho'(t) &= 2(u''(t), m \cdot \nabla u(t)) + 2(u'(t), m \cdot \nabla u'(t)) \\ &\quad +(n-1)(u''(t), u(t)) + (n-1)|u'(t)|^2. \end{aligned} \tag{4.5}$$

Substituting $u'' = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(a(x,t)\frac{\partial u}{\partial x_j}\Big) - \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i}$ into (4.5) yields

$$\begin{aligned} \rho'(t) = 2\Big(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(a(x,t)\frac{\partial u}{\partial x_j}(t)\Big), m \cdot \nabla u(t)\Big) - 2\sum_{i=1}^{n} \Big(\frac{\partial \theta}{\partial x_i}(t), m \cdot \nabla u(t)\Big) + \\ 2\left(u'(t), m \cdot \nabla u'(t)\right) + (n-1)\Big(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(a(x,t)\frac{\partial u}{\partial x_j}\Big), u(t)\Big) - \\ (n-1)\sum_{i=1}^{n} \Big(\frac{\partial \theta}{\partial x_i}(t), u(t)\Big) + (n-1)|u'(t)|^2. \end{aligned} \tag{4.6}$$

Now, our goal is to bound each term of the right-hand side of (4.6). As $m \cdot \nabla u = \sum_{k=1}^{n} m_k \frac{\partial u}{\partial x_k}$ for a moment we will use in the first term of (4.6) the following notation :

$$2\Big(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(a(x,t)\frac{\partial u}{\partial x_j}(t)\Big), m \cdot \nabla u(t)\Big) = 2\Big(\frac{\partial}{\partial x_j}\Big(a\frac{\partial u}{\partial x_j}\Big), m_k \frac{\partial u}{\partial x_k}\Big).$$

In the following steps we will use the Green's formula several time and the boundary conditions (1.3)-(1.5).

Step 1 <u>The first term of (4.6)</u>:

$$\begin{aligned} 2\Big(\frac{\partial}{\partial x_j}\Big(a\frac{\partial u}{\partial x_j}\Big), m_k \frac{\partial u}{\partial x_k}\Big) = -2\int_\Omega a \frac{\partial u}{\partial x_j}\frac{\partial m_k}{\partial x_j}\frac{\partial u}{\partial x_k}dx - \\ 2\int_\Omega a\frac{\partial u}{\partial x_j} m_k \frac{\partial^2 u}{\partial x_j \partial x_k}dx + 2\int_\Gamma a \frac{\partial u}{\partial x_j}\nu_j m_k \frac{\partial u}{\partial x_k} d\Gamma. \end{aligned} \tag{4.7}$$

Modifying some terms of (4.7) yields

$$-2\int_\Omega a\frac{\partial u}{\partial x_j}m_k\frac{\partial^2 u}{\partial x_j\partial x_k}dx = 2\int_\Omega \frac{\partial}{\partial x_k}\Big(a\frac{\partial u}{\partial x_j}m_k\Big)\frac{\partial u}{\partial x_k}dx-$$
$$2\int_\Gamma a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma = 2\int_\Omega \frac{\partial}{\partial x_k}a\frac{\partial u}{\partial x_j}m_k\frac{\partial u}{\partial x_j}dx + 2\int_\Omega a\frac{\partial^2 u}{\partial x_j\partial x_k}m_k\frac{\partial u}{\partial x_j}dx+$$
$$2\int_\Omega a\frac{\partial u}{\partial x_j}\frac{\partial m_k}{\partial x_k}\frac{\partial u}{\partial x_k}dx - 2\int_\Gamma a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma.$$

Using the hypothesis (4.1) yields

$$-2\int_\Omega a\frac{\partial u}{\partial x_j}m_k\frac{\partial^2 u}{\partial x_j\partial x_k}dx \le \kappa\gamma\int_\Omega\Big(\frac{\partial u}{\partial x_j}\Big)^2dx + \int_\Omega a\frac{\partial u}{\partial x_j}\frac{\partial m_k}{\partial x_k}\frac{\partial u}{\partial x_k}dx - \int_\Gamma a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_j}dx.$$

Substituting the last inequality into (4.7) yields

$$2\Big(\frac{\partial}{\partial x_j}\Big(a\frac{\partial u}{\partial x_j}\Big), m_k\frac{\partial u}{\partial x_k}\Big) \le -2\int_\Omega a\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j}dx + \kappa\gamma\int_\Omega\Big(\frac{\partial u}{\partial x_j}\Big)^2dx+$$
$$n\int_\Omega a\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}dx - \int_\Gamma a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}dx + 2\int_\Gamma a\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma.$$

Adding the last inequality from $j,\, k = 1, \ldots, n$ we have

$$2\Big(\sum_{j=1}^n\frac{\partial}{\partial x_j}\Big(a\frac{\partial u}{\partial x_j}\Big), m\cdot\nabla u\Big) \le -(2-n-\kappa)a(t,u,u)-$$
$$\sum_{j,\,k=1}^n\int_\Gamma a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma + 2\sum_{j,\,k=1}^n\int_\Gamma a\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma. \tag{4.8}$$

Step 2 The second term of (4.6):

$$\Big|-2\sum_{i=1}^n\Big(\frac{\partial\theta}{\partial x_i}, m\cdot\nabla u\Big)\Big| \le 2\sum_{i,\,k=1}^n\int_\Omega\Big|\frac{\partial\theta}{\partial x_i}\Big|\,|m_k|\,\Big|\frac{\partial u}{\partial x_k}\Big|dx \le$$
$$\sum_{i=1}^n\frac{[R(x^0)]^2}{\gamma}\int_\Omega\Big|\frac{\partial\theta}{\partial x_i}\Big|^2dx + \sum_{k=1}^n\frac{\gamma}{4}\int_\Omega\Big|\frac{\partial u}{\partial x_k}\Big|^2dx = \frac{[R(x^0)]^2}{\gamma}||\theta||_V^2 + \frac{\gamma}{4}||u||_V^2. \tag{4.9}$$

Step 3 The third term of (4.6):

$$2\,(u', m\cdot\nabla u') = \sum_{k=1}^n\int_\Omega m_k\frac{\partial[u']^2}{\partial x_k}dx = -\sum_{k=1}^n\int_\Omega\frac{\partial m_k}{\partial x_k}[u']^2dx+$$
$$\sum_{k=1}^n\int_\Gamma m_k\nu_k[u']^2d\Gamma = -n|u'|^2 + \int_{\Gamma_1} m\cdot\nu[u']^2d\,\Gamma. \tag{4.10}$$

Step 4 The fourth term of (4.6):

$$(n-1)\Big(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\Big(a\frac{\partial u}{\partial x_j}\Big),u\Big) = -(n-1)\sum_{j=1}^{n}\int_{\Omega}a\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j}dx+$$
$$(n-1)\sum_{j=1}^{n}\int_{\Gamma}a\frac{\partial u}{\partial x_j}\nu_j u d\,\Gamma = -(n-1)a(t,u,u)+(n-1)\int_{\Gamma_1}\frac{\partial u}{\partial \nu_{\mathcal{A}}}u d\,\Gamma$$
$$= -(n-1)a(t,u,u)-(n-1)\int_{\Gamma_1} m\cdot\nu u' u d\Gamma.$$

Let $c_1>0$ be a real number such that $\displaystyle\int_{\Gamma_1}(m\cdot\nu)v^2d\Gamma\le c_1||v||_V^2$ then

$$-(n-1)\int_{\Gamma_1}(m\cdot\nu)u'ud\Gamma \le \frac{n-1}{2}\int_{\Gamma_1}(n-1)(m\cdot\nu)\frac{2c_1}{\gamma}[u']^2d\Gamma+$$
$$\frac{n-1}{2}\int_{\Gamma_1}\frac{\gamma}{n-1}\frac{(m\cdot\nu)}{2c_1}[u]^2d\Gamma \le \frac{c_1(n-1)^2}{\gamma}\int_{\Gamma_1} m\cdot\nu[u']^2d\Gamma+\frac{\gamma}{4}||u||^2\le$$
$$\frac{c_1(n-1)^2}{\gamma}\int_{\Gamma_1} m\cdot\nu[u']^2d\Gamma+\frac{1}{4}a(t,u,u).$$

Thus,

$$(n-1)\Big(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\Big(a(x,t)\frac{\partial u}{\partial x_j}\Big),u\Big)\le -(n-\frac{5}{4})a(t,u,u)+\frac{c_1(n-1)^2}{\gamma}\int_{\Gamma_1} m\cdot\nu[u']^2d\Gamma. \tag{4.11}$$

Step 5 The fifth term of (4.6): Let $\delta_0>0$ be a real number such that $|v|^2\le \delta_0\|v\|^2$ for all $v\in V$. Thus, hence,

$$-(n-1)\sum_{i=1}^{n}\Big(\frac{\partial\theta}{\partial x_i},u\Big)\le(n-1)\sum_{i=1}^{n}\int_{\Omega}\Big(\frac{2\delta_0 n(n-1)}{\gamma}\Big)^{1/2}\Big|\frac{\partial\theta}{\partial x_i}\Big|\Big(\frac{\gamma}{2\delta_0 n(n-1)}\Big)^{1/2}|u|dx\le$$
$$\frac{\delta_0 n(n-1)^2}{\gamma}||\theta||^2+\frac{\gamma}{4\delta_0}|u|^2\le\frac{\delta_0 n(n-1)^2}{\gamma}||\theta||^2+\frac{\gamma}{4}||u||^2. \tag{4.12}$$

Taking into account (4.8)-(4.12) into (4.6) yields

$$\rho'(t)\le-\Big(\frac{1}{4}-\kappa\Big)a(t,u,u)+\Big(\frac{[R(x^0)]^2}{\gamma}+\frac{\delta_0 n(n-1)^2}{\gamma}\Big)||\theta||^2-$$
$$|u'|^2+\Big(1+\frac{c_1(n-1)^2}{\gamma}\Big)\int_{\Gamma_1}(m\cdot\nu)[u']^2d\Gamma- \tag{4.13}$$
$$\sum_{j,k=1}^{n}\int_{\Gamma}a(x,t)\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma+2\sum_{j,k=1}^{n}\int_{\Gamma}a(x,t)\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma.$$

Our next task is to analyze the two last boundary terms in (4.13). Note that on Γ_0 we have

$$\frac{\partial u}{\partial x_j}\nu_j = \frac{\partial u}{\partial \nu} \quad \text{which implies} \quad m\cdot\nabla u = (m\cdot\nu)\frac{\partial u}{\partial \nu} \quad \text{and} \quad |\nabla u|^2 = \Big(\frac{\partial u}{\partial \nu}\Big)^2.$$

Thus, we can write

$$-\sum_{j,k=1}^{n}\int_{\Gamma} a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma = -\sum_{j,k=1}^{n}\int_{\Gamma_1} a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma -$$
$$\sum_{j,k=1}^{n}\int_{\Gamma_0} a\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma = -\int_{\Gamma_1} a\ m\cdot\nu|\nabla u|^2 d\Gamma - \int_{\Gamma_1} a\ m\cdot\nu\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\Gamma,$$

and

$$2\sum_{j,k=1}^{n}\int_{\Gamma} a\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma = 2\sum_{j,k=1}^{n}\int_{\Gamma_1} a\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma + 2\sum_{j,k=1}^{n}\int_{\Gamma_0} a\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma =$$
$$2\int_{\Gamma_1}\frac{\partial u}{\partial \nu_{\mathcal{A}}}(m\cdot\nabla u)\,d\Gamma + 2\int_{\Gamma_0} a\ m\cdot\nu\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\Gamma =$$
$$-2\int_{\Gamma_1}(m\cdot\nu)u^{'}(m\cdot\nabla u)d\Gamma + 2\int_{\Gamma_0} a\ m\cdot\nu\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\Gamma.$$

From definition of Γ_0 it follows $\int_{\Gamma_0} a(x,t)(m\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 \le 0$. Thus,

$$-\sum_{j,k=1}^{n}\int_{\Gamma} a(x,t)\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma + 2\sum_{j,k=1}^{n}\int_{\Gamma} a(x,t)\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma =$$
$$-2\int_{\Gamma_1}(m\cdot\nu)u'(m\cdot\nabla u)d\Gamma - \int_{\Gamma_1} a(x,t)(m\cdot\nu)|\nabla u|^2 d\Gamma. \tag{4.14}$$

On the other hand,

$$-2\int_{\Gamma_1}(m\cdot\nu)u'(m\cdot\nabla u)d\Gamma \le 2R(x^0)\int_{\Gamma_1}(a_0)^{1/2}(m\cdot\nu)^{1/2}|u'|(a_0)^{1/2}(m\cdot\nu)^{1/2}|\nabla u|d\Gamma \le$$
$$\frac{R(x^0)}{a_0}\int_{\Gamma_1}(m\cdot\nu)|u'|^2 d\Gamma + \int_{\Gamma_1} a(x,t)(m\cdot\nu)|\nabla u|^2 d\Gamma. \tag{4.15}$$

From (4.14) and (4.15) yields

$$-\sum_{j,k=1}^{n}\int_{\Gamma} a(x,t)\frac{\partial u}{\partial x_j}\nu_k m_k\frac{\partial u}{\partial x_k}d\Gamma + 2\sum_{j,k=1}^{n}\int_{\Gamma} a(x,t)\frac{\partial u}{\partial x_j}\nu_j m_k\frac{\partial u}{\partial x_k}d\Gamma \le \frac{R(x^0)}{a_0}\int_{\Gamma_1}(m\cdot\nu)[u']^2 d\Gamma. \tag{4.16}$$

Substituting (4.15) into (4.13) yields

$$\rho'(t) \le -\Big(\frac{1}{4}-\kappa\Big)a(t,u,u) + \Big(\frac{[R(x^0)]^2}{\gamma} + \frac{n(n-1)^2}{2\gamma\delta_0}\Big)||\theta||^2 -$$
$$|u'|^2 + \Big(1 + \frac{c_1(n-1)^2}{\gamma} + \frac{R(x^0)}{a_0}\Big)\int_{\Gamma_1}(m\cdot\nu)[u']^2 d\Gamma. \tag{4.17}$$

Multiplying $\rho'(t)$ by $\varepsilon > 0$ it follows from (4.2) and (4.17) that

$$\frac{d}{dt}E(t) + \int_{\Gamma_1} \Big\{\alpha_0 - \Big[\frac{\epsilon}{2} + \varepsilon\Big(1 + \frac{c_1(n-1)^2}{\gamma} + \frac{R(x^0)}{a_0}\Big)\Big]\Big\}[u']^2 d\Gamma \leq$$
$$-\varepsilon\Big(\frac{1}{4} - \kappa\Big)a(t,u,u) - \Big[\gamma - \varepsilon\Big(\frac{[R(x^0)]^2}{\gamma} + \frac{n(n-1)^2}{2\gamma\delta_0}\Big)\Big]||\theta||^2$$
$$-\varepsilon|u'|^2 \leq -\varepsilon\Big(\frac{1}{4} - \kappa\Big)a(t,u,u) - \frac{1}{\delta_0}\Big[\gamma - \varepsilon\Big(\frac{[R(x^0)]^2}{\gamma} + \frac{n(n-1)^2}{2\gamma\delta_0}\Big)\Big]|\theta|^2 - \varepsilon|u'|^2. \tag{4.18}$$

Let τ be a real positive number defined by

$$\tau = \min\Big\{\varepsilon\Big(\frac{1}{4} - \kappa\Big),\ \ \frac{1}{\delta_0}\Big[\gamma - \varepsilon\Big(\frac{R^2}{\gamma} + \frac{n(n-1)^2}{2\gamma\delta_0}\Big)\Big],\ \ \varepsilon\Big\}.$$

Hence, (4.17) and (4.18) it follows that

$$\frac{d}{dt}E_\varepsilon(t) + \int_{\Gamma_1} \Big\{\alpha_0 - \Big[\frac{\epsilon}{2} + \varepsilon\Big(1 + \frac{c_1(n-1)^2}{\gamma} + \frac{R(x^0)}{a_0}\Big)\Big]\Big\}[u']^2 d\Gamma \leq -\tau E(t), \tag{4.19}$$

where $E_\varepsilon(t) = E(t) + \varepsilon\rho(t)$.

It is easy to prove that there are positive constants η_1 and η_2 such that

$$\eta_1 E(t) \leq E_\varepsilon(t) \leq \eta_2 E(t) \tag{4.20}$$

for all $t \geq 0$ and for suitable ε. From (4.19) and (4.20) we obtain (4.3). Therefore, the theorem 4.1 is established ■

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