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Approximate Approximations for the Poisson and the Stokes Equations

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Abstract

The method of approximate approximations is based on generating functions representing an approximate partition of the unity, only. In the present paper this method is used for the numerical solution of the Poisson equation and the Stokes system in $\mathbb{R}^n$ $(n = 2, 3)$. The corresponding approximate volume potentials will be computed explicitly in these cases, containing a one-dimensional integral, only. Numerical simulations show the efficiency of the method and confirm the expected convergence of essentially second order, depending on the smoothness of the data.

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1 Introduction

In 1991 V. Maz'ya introduced an approximation method, called the method of approximate approximations [3]. Here a given function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is approximated by a linear combination f_h $(h > 0)$ of radial smooth exponentially decreasing basic functions (compare [6], [7], [8]). In contrast to a linear combination of splines this system of basic functions leads only to an approximate partition of the unity. Hence the approximation procedure does not converge as $h \to 0$. For practical computations, however, this lack of convergence does not play an important role, since the error between f and its approximation f_h can be controlled via a certain parameter and hence chosen to be of the same magnitude as the computer accuracy. Furthermore, the method of approximate approximations has great advantages for the numerical solution of Cauchy problems of the form $Du = f$, where D is a suitable linear partial differential operator in $\mathbb{R}^n$. In some cases explicit formulas for the approximate volume potentials can be developed if the right hand side f is approximated by f_h. In these formulas, instead of a multi-dimensional integration, often there remains a one-dimensional integral only, for instance an expression containing the error function (see [2], [5]). Recently, the method of approximate approximations has also been applied successfully for the numerical treatment of boundary value problems (see [9], [10]).

In the present paper the method of approximate approximations is carried out explicitly for two important Cauchy problems in $\mathbb{R}^n$ $(n = 2, 3)$, the Poisson equation $-\Delta v = f$ and the Stokes system $-\Delta u + \nabla p = f$, $\operatorname{div} u = 0$, the latter well-known from hydrodynamics. In Section 2 the method is motivated and introduced for the approximation of functions given on the real line. Here also error estimates are presented (Lemma 1). In Section 3 the method is applied to the Poisson equation. Here, using two different approaches, explicit expressions for the corresponding approximate volume potentials in two and three dimensions are given, containing the exponential integral function and the error function, respectively (Theorem 2 and Theorem 3). In Section 4 explicit expressions for the solution of the Stokes system are given, where even in this case the approximate velocity potentials depend on the above mentioned functions, only (Theorem 4 and Theorem 5). In Section 5 and 6 numerical simulations for both the Poisson equation and the Stokes system $(n = 2)$ are carried out, where here in addition the smoothness of the density function can be controlled by some parameter. In all cases the numerical simulations show essential convergence of second order, as expected from the error estimates.

2 Approximate approximations on $\mathbb{R}$

We consider the Gaussian probability function $\varphi_{\mu,\sigma}$ of the normal distribution with mean μ and variance σ^2, defined by

$$\varphi_{\mu,\sigma}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\mu - x)^2}{2\sigma^2}\right). \tag{1}$$

It is well–known that this function takes it's maximum at $x = \mu$ and has two turning points at $x = \mu \pm \sigma$, such that the variance σ^2 somehow represents a measure of the Gaussian bell. Since $\varphi_{\mu,\sigma}$ is a probability density on the real line we have

$$\int_{-\infty}^{+\infty} \varphi_{\mu,\sigma}(x)dx = 1. \tag{2}$$

Replacing integration by a simple quadrature rule we obtain

$$\sum_{k\in\mathbb{Z}} \varphi_{\mu,\sigma}(k) \approx 1.$$

Let us consider the left–hand side as a function of μ, i.e.

$$\mu \mapsto \Phi_\sigma(\mu) := \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu-k)^2}{2\sigma^2}\right). \tag{3}$$

We investigate the difference between Φ_σ and the constant 1. Since Φ_σ is an even function having the period $p = 1$ we obtain the Fourier series expansion

$$\Phi_\sigma(\mu) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(2m\pi\mu), \quad |\mu| < \tfrac{1}{2}.$$

An easy calculation leads to the Fourier coefficients

$$a_m = 2\exp\left(-2\sigma^2 m^2\pi^2\right), \quad m \in \mathbb{N}_0.$$

It follows

$$\Phi_\sigma(\mu) - 1 = 2\sum_{k=1}^{\infty} \exp\left(-2\sigma^2 k^2\pi^2\right)\cos(2k\pi\mu),$$

and this implies

$$|\Phi_\sigma(\mu) - 1| \;\leq\; 2\sum_{k=1}^{\infty} \exp\left(-2\sigma^2 k^2\pi^2\right) \approx \begin{cases} 10^{-2}\,, & \sigma = \frac{1}{2}, \\ 10^{-9}\,, & \sigma = 1, \\ 10^{-34}, & \sigma = 2. \end{cases} \tag{4}$$

Analogously, for the derivatives

$$\begin{aligned} \Phi_\sigma'(\mu) &:= -4\pi \sum_{k=1}^{\infty} k\exp\left(-2\sigma^2 k^2\pi^2\right)\sin(2k\pi\mu), \\ \Phi_\sigma''(\mu) &:= -8\pi^2 \sum_{k=1}^{\infty} k^2\exp\left(-2\sigma^2 k^2\pi^2\right)\cos(2k\pi\mu) \end{aligned}$$

we find

$$|\Phi'_\sigma(\mu)| \approx \begin{cases} 10^{-1}\,, & \sigma = \frac{1}{2}, \\ 10^{-6}\,, & \sigma = 1, \\ 10^{-34}, & \sigma = 2, \end{cases} \qquad \text{and} \qquad |\Phi''_\sigma(\mu)| \approx \begin{cases} 10^{-1}\,, & \sigma = \frac{1}{2}, \\ 10^{-7}\,, & \sigma = 1, \\ 10^{-33}, & \sigma = 2. \end{cases}$$

In the following, let us assume $\sigma := 1$. In contrast to splines, compare e.g. the piecewise linear B–splines in Figure 1, the function

$$\Phi(\mu) := \Phi_1(\mu) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \exp\left(-\frac{(\mu - k)^2}{2} \right) \tag{5}$$

generates only an approximate partition of the unity (see Figure 2).

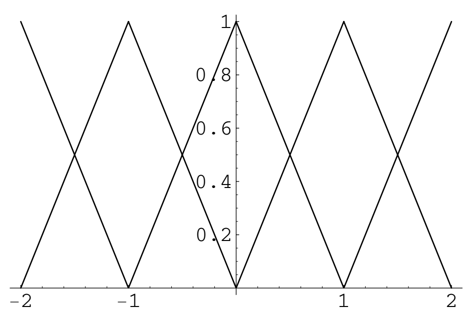

Figure 1: Exact partition of the unity

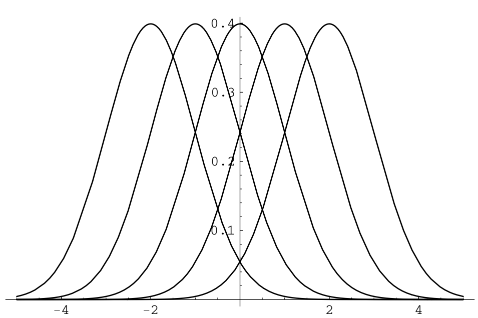

Figure 2: Approximate partition of the unity

Now let us use the function (5) for the approximation of a given function

$f:\mathbb{R}\to\mathbb{R}$. For this purpose we choose $h>0$ and define

$$f_h(x) := \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-hk}{h}\right)^2\right) f(hk). \tag{6}$$

Since we are using an approximate partition of the unity, only, we cannot expect convergence of the resulting sequence if h tends to zero. Anyhow, let us study the error

$$\varepsilon_h(x) := f_h(x) - f(x)$$

for $h\to 0$ assuming a certain regularity on f. To do so we need the space $C_b^m(\mathbb{R})$ of functions having bounded continuous derivatives on $\mathbb{R}$ up to the order $m\in\mathbb{N}$.

Lemma 1 *Let $f\in C_b^2(\mathbb{R})$, $h>0$, and f_h defined by (6). Then the error $\varepsilon_h(x)$ satisfies in $x\in\mathbb{R}$ the following estimate:*

$$|\varepsilon_h(x)| \le \frac{h^2}{2}\,\|f''\|_\infty\Big(\left|\Phi\left(\tfrac{x}{h}\right)\right| + \left|\Phi''\left(\tfrac{x}{h}\right)\right|\Big) + h\,|f'(x)|\left|\Phi'\left(\tfrac{x}{h}\right)\right| + |f(x)|\left|\Phi\left(\tfrac{x}{h}\right)-1\right|.$$

Here Φ is the function defined by (5) and $\|f\|_\infty := \sup_{x\in\mathbb{R}}|f(x)|$ is the norm in $L^\infty(\mathbb{R})$.

Proof: We use the decomposition

$$\begin{aligned}
\varepsilon_h(x) &= f_h(x) - f(x)\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{1}{2}\left(\frac{x}{h}-k\right)^2\right)(f(hk)-f(x)) \;+\; f(x)\left(\Phi\left(\tfrac{x}{h}\right)-1\right)\\
&=: S_1(x) + S_2(x).
\end{aligned}$$

By Taylor expansion we have

$$f(hk) - f(x) = (hk-x)f'(x) + \frac{(hk-x)^2}{2}f''(\xi_h),$$

where $\xi_h\in\mathbb{R}$ denotes some point between x and hk. This implies

$$\begin{aligned}
S_1(x) &= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{(x-hk)^2}{2h^2}\right)(hk-x)f'(x)\\
&+ \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{(x-hk)^2}{2h^2}\right)\frac{(hk-x)^2}{2}f''(\xi_h) =: s_1(x)+s_2(x).
\end{aligned}$$

Due to

$$\Phi'\left(\tfrac{x}{h}\right) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(x-hk)^2}{2h^2}\right) \tfrac{hk-x}{h}$$

we obtain

$$s_1(x) = hf'(x)\Phi'\left(\tfrac{x}{h}\right).$$

Using

$$\Phi''\left(\tfrac{x}{h}\right) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(x-hk)^2}{2h^2}\right) \frac{(hk-x)^2}{h^2} - \Phi\left(\tfrac{x}{h}\right),$$

for the second summand it follows

$$s_2(x) = \frac{h^2}{2} f''(\xi_h) \left(\Phi''\left(\frac{x}{h}\right) + \Phi\left(\frac{x}{h}\right)\right).$$

This proves the lemma. □

The estimate of Lemma 1 shows that we are using an approximation essentially of second order, since in practise only the term

$$\frac{h^2}{2} \|f''\|_\infty \left|\Phi\left(\tfrac{x}{h}\right)\right|$$

has to be taken into account, all other factors are neglectably small. Therefore the expression approximate approximation seems to be reasonable (compare [3]).

The method caries over immediately to the n–dimensional case, where a given function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated by

$$f_h(x) := \frac{1}{\sqrt{(2\pi)^n}} \sum_{k\in\mathbb{Z}^n} \exp\left(-\tfrac{1}{2}\left|\tfrac{x-hk}{h}\right|^2\right) f(hk). \tag{7}$$

All the above statements hold true in this case, too.

3 Application to the Poisson equation

To use this approximation method for the numerical solution of the Poisson equation

$$-\triangle v = f \quad \text{in } \mathbb{R}^n \qquad (n = 2, 3) \tag{8}$$

we proceed as follows: It is well–known [1] that a solution of (8) is given by the volume potential

$$Vf(x) := \int_{\mathbb{R}^n} e(x-y) f(y)\, dy \qquad (n=2,3). \tag{9}$$

Here

$$e(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x|}, & n=2, \\ \frac{1}{4\pi} \frac{1}{|x|} \quad , & n=3 \end{cases} \tag{10}$$

denotes the fundamentiel solution of the Laplacian in $\mathbb{R}^n$.

To approximate the volume potential Vf we replace the given function f by the approximation f_h, defined by (7). This leads to an approximate solution of (8) in the form

$$\begin{aligned} v_h(x) &:= Vf_h = \int_{\mathbb{R}^n} e(x-y) \cdot \frac{1}{\sqrt{(2\pi)^n}} \sum_{m\in\mathbb{Z}^n} \exp\left(-\frac{1}{2}\left|\frac{y}{h}-m\right|^2\right) f(hm)\, dy \\ &= \sum_{m\in\mathbb{Z}^n} S_{m,h}(x) f(hm) \end{aligned} \tag{11}$$

with

$$S_{m,h}(x) := \begin{cases} \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^2} \ln \frac{1}{|x-y|} \cdot \exp\left(-\frac{1}{2}\left|\frac{y}{h}-m\right|^2\right) dy \quad , & n=2, \\ \frac{1}{4\pi\sqrt{(2\pi)^3}} \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|} \cdot \exp\left(-\frac{1}{2}\left|\frac{y}{h}-m\right|^2\right) dy, & n=3. \end{cases}$$

The weights $S_{m,h}(x)$ can be determined analytically:

Theorem 2 *Let $\xi := x/h - m$. Then for $n=2$ we have*

$$S_{m,h}(x) = -\frac{h^2}{4\pi}\left\{\ln(2h^2) - C + \operatorname{exint}\left(\frac{1}{2}|\xi|^2\right)\right\}.$$

Here $C = 0,577216\ldots$ is Euler's constant and the function exint is defined by

$$\operatorname{exint}(x) := \int\limits_0^x \frac{1-\exp(-t)}{t}\, dt.$$

Proof: Substituting $z := y/h - m$, due to $dy = h^2\, dz$ we have

$$\begin{aligned} S_{m,h}(x) &= -\frac{1}{4\pi^2} h^2 \int_{\mathbb{R}^2} \ln(h\,|\xi - z|) \exp\left(-\frac{|z|^2}{2}\right) dz \\ &= -\frac{1}{4\pi^2} h^2 \int_{\mathbb{R}^2} (\ln h) \exp\left(-\frac{|z|^2}{2}\right) dz - \frac{1}{4\pi^2} h^2 \int_{\mathbb{R}^2} \ln(|\xi - z|) \exp\left(-\frac{|z|^2}{2}\right) dz \\ &=: -\frac{1}{4\pi^2}\left(S_1(\xi) + S_2(\xi)\right). \end{aligned}$$

Using polar coordinates and the substitution $t = r^2/2$ for the first term we obtain

$$\begin{aligned} S_1(\xi) &= h^2 \int_{\mathbb{R}^2} (\ln h) \exp\left(-\frac{|z|^2}{2}\right) dz \\ &= 2\pi h^2 \ln h. \end{aligned}$$

Since $S_2(\xi)$ depends only on $|\xi|$, a similar calculation leads to

$$\begin{aligned} S_2(\xi) &= h^2 \int_{\mathbb{R}^2} \ln|\xi - z| \exp\left(-\frac{|z|^2}{2}\right) dz \\ &= h^2 \int_0^\infty \int_0^{2\pi} \ln\sqrt{|\xi|^2 + r^2 - 2\,|\xi| r\cos\theta}\, d\theta \exp\left(-\frac{r^2}{2}\right) r\, dr \\ &= h^2 \int_0^\infty 2\pi \ln\left(\max\{|\xi|, r\}\right) \exp\left(-\frac{r^2}{2}\right) r\, dr. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} S_2(\xi) &= 2\pi h^2 \int_0^{|\xi|} \ln|\xi| \exp\left(-\frac{r^2}{2}\right) r\, dr + 2\pi h^2 \int_{|\xi|}^\infty \ln r \exp\left(-\frac{r^2}{2}\right) r\, dr \\ &= \pi h^2 \left(\ln 2 - C + \operatorname{exint}\left(\frac{|\xi|^2}{2}\right)\right), \end{aligned}$$

and this proves the assertion.

□

To prove an analogous formula for $n = 3$ we use a different method.

Theorem 3 *Let $\xi := x/h - m$. Then for $n = 3$ we have*

$$S_{m,h}(x) = \frac{h^2}{\sqrt{(2\pi)^3}} \frac{1}{|\xi|} \int_0^{|\xi|} \exp\left(-\frac{t^2}{2}\right) dt.$$

Proof: Substituting $z := y/h - m$, due to $dy = h^2\, dz$ we have

$$S_{m,h}(x) = \frac{h^2}{4\pi\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{1}{|\xi - z|} \cdot \exp\left(-\tfrac{1}{2}\,|z|^2\right) dz \;=:\; \frac{h^2}{\sqrt{(2\pi)^3}} \cdot v(\xi).$$

Using spherical coordinates we can show that $v(\xi) = v(r, \theta, \varphi)$ depends only on $r = |\xi|$.

The integral $v(\xi)$ is a volume potential that solves (8) with $f = \exp\left(-\frac{1}{2}|\xi|^2\right)$, i.e.

$$\begin{aligned}
-\exp\left(-\tfrac{1}{2}|\xi|^2\right) &= \triangle v(\xi) = \triangle v(r,\theta,\varphi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) \qquad \left(\text{due to } \frac{\partial v}{\partial\theta} = \frac{\partial v}{\partial\varphi} = 0\right).
\end{aligned}$$

We obtain

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = -r^2\exp\left(-\tfrac{1}{2}r^2\right),$$

i.e.

$$r^2\frac{\partial v}{\partial r} = -\int_{t=0}^{r} t^2\exp\left(-\tfrac{1}{2}t^2\right)\,dt.$$

From this it follows

$$r^2\frac{\partial v}{\partial r} = r\exp\left(-\tfrac{1}{2}r^2\right) - \int_{z=0}^{r}\exp\left(-\tfrac{1}{2}z^2\right)\,dz,$$

i.e.

$$v(\infty) - v(r) = \int_{t=r}^{\infty}\left[\frac{1}{t}\exp\left(-\tfrac{1}{2}t^2\right) - \frac{1}{t^2}\int_{z=0}^{t}\exp\left(-\tfrac{1}{2}z^2\right)\,dz\right]\,dt.$$

We obtain $v(\infty) = 0$, and with partial integration we find

$$\begin{aligned}
v(r) &= -\int_{t=r}^{\infty}\frac{1}{t}\exp\left(-\tfrac{1}{2}t^2\right)\,dt + \int_{t=r}^{\infty}\frac{1}{t^2}\int_{z=0}^{t}\exp\left(-\tfrac{1}{2}z^2\right)\,dz\,dt \\
&= -\int_{t=r}^{\infty}\frac{1}{t}\exp\left(-\tfrac{1}{2}t^2\right)\,dt - \frac{1}{t}\int_{z=0}^{t}\exp\left(-\tfrac{1}{2}z^2\right)\,dz\,\Big|_{t=r}^{\infty} + \int_{t=r}^{\infty}\frac{1}{t}\exp\left(-\tfrac{1}{2}t^2\right)\,dt \\
&= \frac{1}{r}\int_{z=0}^{r}\exp\left(-\tfrac{1}{2}z^2\right)\,dz.
\end{aligned}$$

This proves the asserted formula for $n = 3$, too.

□

4 Application to the Stokes equations

Now we will use the approximation method to solve numerically the Stokes equations

$$\left.\begin{array}{c} -\triangle u + \nabla p = f \\ \operatorname{div} u = 0 \end{array}\right\} \text{ in } \mathbb{R}^n \qquad (n = 2, 3). \tag{12}$$

Here $u := (u_1, ..., u_n)$ is the unknown velocity field and $\nabla p := (\partial_1 p, ..., \partial_n p)$ the unknown pressure gradient of a viscous incompressible fluid flow, and the exernal force density $f := (f_1, ... f_n)$ is given.

In the following we approximate the velocity field u, only, but there exists an analogous method for the pressure p. The corresponding volume potential part

$$u(x) = Vf(x) := \int_{\mathbb{R}^n} E(x-y) \cdot f(y)\, dy \qquad (n = 2, 3), \tag{13}$$

which solves (12), is defined using the fundamential solution $E(x) = E_{ij}(x)$ $(i, j = 1 \dots n)$ of the Stokes operator in $\mathbb{R}^n$, i.e.

$$E_{ij}(x) = \frac{1}{2\omega_n}\left\{\frac{x_i x_j}{|x|^n} + \delta_{ij}\left\{\begin{array}{ll} \ln\frac{1}{|x|}, & n=2 \\ |x|^{-1}, & n=3 \end{array}\right\}\right\}. \tag{14}$$

Here ω_n denotes the surface of the $(n-1)$- dimensional unit ball (see [1]).

To approximate the volume potential Vf we replace each component f_j $(j = 1, \dots, n)$ of the given function f by the approximation (7), i.e. by

$$f_j^h(y) := \frac{1}{\sqrt{(2\pi)^n}} \sum_{m \in \mathbb{Z}^n} \exp\left(-\frac{1}{2}\left|\frac{y-hm}{h}\right|^2\right) f_j(hm).$$

This leads to an approximate solution u of (12) in the form

$$u_h(x) := Vf_h = \int_{\mathbb{R}^n} E(x-y) \cdot f_h(y)\, dy = \sum_{m \in \mathbb{Z}^n} A^{m,h}(x) \cdot f(hm) \tag{15}$$

with $A^{m,h} = A^{m,h}_{ij}$, $(i, j = 1 \dots n)$:

$$A^{m,h}_{ij}(x) := \left\{\begin{array}{ll} \frac{1}{8\pi^2} \int\limits_{\mathbb{R}^2} \left(\delta_{ij} \ln\frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right) \exp\left(-\frac{1}{2}\left|\frac{y}{h} - m\right|^2\right) dy \quad , & n=2, \\ \frac{1}{8\pi\sqrt{(2\pi)^3}} \int\limits_{\mathbb{R}^3} \left(\delta_{ij} \frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right) \exp\left(-\frac{1}{2}\left|\frac{y}{h} - m\right|^2\right) dy, & n=3. \end{array}\right.$$

The weights $A^{m,h}(x)$ can be determined analytically:

Theorem 4 *Let* $\xi := x/h - m$. *Then for* $n = 2$ *and* $i, j = 1, 2$ *and* $|\xi| \neq 0$ *we have*

$$\begin{aligned} A^{m,h}_{ij}(x) &= \frac{1}{8\pi} h^2 \delta_{ij} \left[-\ln(2h^2) + C - \operatorname{exint}\left(\tfrac{1}{2}|\xi|^2\right) + \frac{1 - \exp\left(-\frac{1}{2}|\xi|^2\right)}{\frac{1}{2}|\xi|^2}\right] \\ &+ \frac{1}{8\pi} h^2 \left\{\frac{\xi_i\, \xi_j}{\frac{1}{2}|\xi|^2} - \frac{\xi_i\, \xi_j}{\frac{1}{4}|\xi|^4}\left[1 - \exp\left(-\tfrac{1}{2}|\xi|^2\right)\right]\right\}, \end{aligned}$$

and for $|\xi| = 0$ *we have*

$$A_{ij}^{m,h}(x) = \frac{1}{8\pi} h^2 \delta_{ij} \left(C - \ln\left(2h^2\right) + 1\right),$$

where C *is Euler's constant.*

Proof: By using the functions

$$S(x) := \int\limits_{\mathbb{R}^2} \ln \tfrac{1}{|x-y|} \cdot \exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right) dy, \qquad H(x) := \tfrac{1}{2}|x|^2 \left(\ln|x| - \tfrac{1}{2}\right)$$

with $\frac{\partial^2}{\partial x_i \partial x_j} H(x) = -\delta_{ij} \ln \frac{1}{|x|} + \frac{x_i x_j}{|x|^2}$ for $x \in \mathbb{R}^2$ we obtain

$$\begin{aligned}
A_{ij}^{m,h}(x) &:= \frac{1}{8\pi^2} \int\limits_{\mathbb{R}^2} \left(\delta_{ij} \ln \tfrac{1}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} \right) \cdot \exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right) dy \\
&= \frac{1}{8\pi^2} \left(\int_{\mathbb{R}^2} \frac{\partial^2}{\partial x_i \partial x_j} H(x-y) \cdot \exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right) dy + 2\delta_{ij} S(x) \right) \\
&= \frac{1}{8\pi^2} \left(\frac{\partial^2}{\partial x_i \partial x_j} \left(\int_{\mathbb{R}^2} H(x-y) \cdot \exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right) dy \right) + 2\delta_{ij} S(x) \right) \\
&=: \frac{1}{8\pi^2} \left(\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{2} B(x) + 2\delta_{ij} S(x) \right).
\end{aligned}$$

Following Theorem 2 with $\xi = x/h - m$ we find

$$S(x) = -\pi h^2 \left\{ \ln(2h^2) - C + \operatorname{exint}\left(\tfrac{1}{2}|\xi|^2\right) \right\}.$$

For $B(x)$ with $z = y/h - m$ and $\xi = x/h - m$ it follows

$$\begin{aligned}
B(x) &= h^4 \int_{\mathbb{R}^2} |\xi - z|^2 \ln|\xi - z| \exp\left(-\tfrac{1}{2}|z|^2\right) dz \\
&+ (\ln h - \tfrac{1}{2}) h^4 \int_{\mathbb{R}^2} |\xi - z|^2 \exp\left(-\tfrac{1}{2}|z|^2\right) dz =: h^4 F(x) + (\ln h - \tfrac{1}{2}) h^4 G(x).
\end{aligned}$$

By transformation in polar coordinates (r, φ) finally we obtain

$$G(x) = \int\limits_0^\infty \int\limits_0^{2\pi} \left[|\xi|^2 + r^2 - 2|\xi| r \cos\varphi \right] r \exp\left(-\tfrac{1}{2} r^2\right) d\varphi \, dr = 2\pi |\xi|^2 + 4\pi$$

and

$$\begin{aligned}
F(x) &= \frac{1}{2}\int_0^\infty\int_0^{2\pi}\left[|\xi|^2+r^2-2|\xi|r\cos\varphi\right]\ln\left[|\xi|^2+r^2-2|\xi|r\cos\varphi\right] r\exp\left(-\frac{1}{2}r^2\right)\,d\varphi\,dr\\
&= \frac{1}{2}\int_0^\infty\Bigg[(|\xi|^2+r^2)\int_0^{2\pi}\ln\left[|\xi|^2+r^2-2|\xi|r\cos\varphi\right]\,d\varphi\\
&- 2|\xi|r\int_0^{2\pi}\cos\varphi\ln\left[|\xi|^2+r^2-2|\xi|r\cos\varphi\right]\,d\varphi\Bigg] r\exp\left(-\frac{1}{2}r^2\right)\,dr\\
&= \frac{1}{2}\int_0^\infty\left[(|\xi|^2+r^2)\,S(\xi,r)-2|\xi|r\,T(\xi,r)\right] r\exp\left(-\frac{1}{2}r^2\right)\,dr.
\end{aligned}$$

Since

$$S(\xi,r)=\begin{cases} 4\pi\ln r & , \quad |\xi|\le r\\ 4\pi\ln|\xi| & , \quad r<|\xi|\end{cases}$$

and

$$T(\xi,r)=\begin{cases} -\frac{2\pi|\xi|}{r} & , \quad |\xi|\le r\\ -\frac{2\pi r}{|\xi|} & , \quad r<|\xi|\end{cases}$$

we obtain

$$F(x)=2\pi\left[2-\exp\left(-\tfrac{1}{2}|\xi|^2\right)+\left(\tfrac{1}{2}|\xi|^2+1\right)\cdot\left\{\ln 2-C+\operatorname{exint}\left(\tfrac{1}{2}|\xi|^2\right)\right\}\right]$$

and

$$B(x)=2\pi h^4\left\{2-\exp\left(-\frac{1}{2}|\xi|^2\right)+\left(\frac{1}{2}|\xi|^2+1\right)\left[\ln(2h^2)-C-1+\operatorname{exint}\left(\frac{1}{2}|\xi|^2\right)\right]\right\}.$$

Due to

$$\begin{aligned}
\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{2}B(x) = \pi h^2\Bigg\{&\delta_{ij}\left[\ln\left(2h^2\right)-C+\operatorname{exint}\left(\tfrac{1}{2}|\xi|^2\right)\right]\\
+\;&\delta_{ij}\frac{1-\exp\left(-\frac{1}{2}|\xi|^2\right)}{\frac{1}{2}|\xi|^2}+\frac{\xi_i\,\xi_j}{\frac{1}{2}|\xi|^2}-\frac{\xi_i\,\xi_j}{\frac{1}{4}|\xi|^4}\left[1-\exp\left(-\tfrac{1}{2}|\xi|^2\right)\right]\Bigg\},
\end{aligned}$$

for $|\xi|\neq 0$ we find

$$\begin{aligned}
A_{ij}^{m,h}(x) &= \frac{1}{8\pi^2}\left\{\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{2}B(x)+2\delta_{ij}S(x)\right\}\\
&= \frac{1}{8\pi}h^2\delta_{ij}\left[-\ln(2h^2)+C-\operatorname{exint}\left(\tfrac{1}{2}|\xi|^2\right)+\frac{1-\exp\left(-\frac{1}{2}|\xi|^2\right)}{\frac{1}{2}|\xi|^2}\right]\\
&+ \frac{1}{8\pi}h^2\left\{\frac{\xi_i\,\xi_j}{\frac{1}{2}|\xi|^2}-\frac{\xi_i\,\xi_j}{\frac{1}{4}|\xi|^4}\left[1-\exp\left(-\tfrac{1}{2}|\xi|^2\right)\right]\right\},
\end{aligned}$$

and for $|\xi| = 0$

$$A_{ij}^{m,h}(x) = \delta_{ij}\frac{1}{8\pi}h^2\left\{C - \ln\left(2h^2\right) + 1\right\}.$$

□

In an analogous way we can prove

Theorem 5 *Let $\xi := x/h - m$. Then for $n = 3$ and $i, j = 1, \ldots 3$ and $|\xi| \neq 0$ we have*

$$\begin{aligned} A_{ij}^{m,h}(x) &= \frac{1}{2\sqrt{(2\pi)^3}}\delta_{ij}h^2\frac{1}{|\xi|^2}\left\{-\exp\left(-\frac{|\xi|^2}{2}\right) + \left(|\xi| + \frac{1}{|\xi|}\right)\int\limits_0^{|\xi|}\exp\left(-\frac{t^2}{2}\right)\,dt\right\} \\ &+ \frac{1}{2\sqrt{(2\pi)^3}}h^2\frac{\xi_i\xi_j}{|\xi|^4}\left\{3\exp\left(-\frac{|\xi|^2}{2}\right) + \left(|\xi| - \frac{3}{|\xi|}\right)\int\limits_0^{|\xi|}\exp\left(-\frac{t^2}{2}\right)\,dt\right\}, \end{aligned}$$

and for $|\xi| = 0$ we have

$$A_{ij}^{m,h}(x) = \frac{2}{3\sqrt{(2\pi)^3}}\delta_{ij}h^2.$$

Proof: By using the functions

$$S(x) := \int\limits_{\mathbb{R}^3}\frac{1}{|x-y|}\cdot\exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right)\,dy, \qquad H(x) := |x|$$

with $\frac{\partial^2}{\partial x_i\partial x_j}H(x) = \delta_{ij}\frac{1}{|x|} - \frac{x_ix_j}{|x|^3}$ for $x \in \mathbb{R}^3$ we obtain

$$\begin{aligned} A_{ij}^{m,h}(x) &:= \frac{1}{8\pi}\frac{1}{\sqrt{(2\pi)^3}}\int\limits_{\mathbb{R}^3}\left(\delta_{ij}\frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right)\cdot\exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right)\,dy \\ &= \frac{1}{8\pi}\frac{1}{\sqrt{(2\pi)^3}}\left(\int_{\mathbb{R}^3}2\delta_{ij}S(x) - \frac{\partial^2}{\partial x_i\partial x_j}H(x-y)\cdot\exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right)\,dy\right) \\ &= \frac{1}{8\pi}\frac{1}{\sqrt{(2\pi)^3}}\left(2\delta_{ij}S(x) - \frac{\partial^2}{\partial x_i\partial x_j}\left(\int_{\mathbb{R}^2}H(x-y)\cdot\exp\left(-\tfrac{1}{2}\left|\tfrac{y}{h} - m\right|^2\right)\,dy\right)\right) \\ &=: \frac{1}{8\pi}\frac{1}{\sqrt{(2\pi)^3}}\left(2\delta_{ij}S(x) - \frac{\partial^2}{\partial x_i\partial x_j}B(x)\right). \end{aligned}$$

Using Theorem 3 with $\xi = x/h - m$ we find

$$S(x) = 4\pi h^2\frac{1}{|\xi|}\int\limits_0^{|\xi|}\exp\left(-\tfrac{1}{2}t^2\right)\,dt.$$

For $B(x)$ with $z = y/h - m$ and $\xi = x/h - m$ it follows

$$B(x) = h^4 \int_{\mathbb{R}^3} |\xi - z| \exp\left(-\tfrac{1}{2}|z|^2\right) dz.$$

Using polar coordinates (r, θ, φ) we can show that B depends only on $|\xi|$, and we obtain

$$\begin{aligned} B(x) &= h^4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \sqrt{|\xi|^2 + r^2 - 2|\xi| r \cos\theta} \exp\left(-\tfrac{1}{2}r^2\right) r^2 \sin\theta \, d\varphi \, d\theta \, dr \\ &= 2\pi h^4 \int_{r=0}^{\infty} \int_{t=-1}^{1} \sqrt{|\xi|^2 + r^2 - 2|\xi| r t} \exp\left(-\tfrac{1}{2}r^2\right) r^2 \, dt \, dr. \end{aligned}$$

For $|\xi| \neq 0$ we find

$$\begin{aligned} B(x) &= \tfrac{2}{3}\pi h^4 \frac{1}{|\xi|} \int_{r=0}^{\infty} r \exp\left(-\tfrac{1}{2}r^2\right) \left[||\xi| + r|^3 - ||\xi| - r|^3 \right] dr \\ &= 4\pi h^4 \left\{ \exp\left(-\tfrac{1}{2}|\xi|^2\right) + \left(|\xi| + \frac{1}{|\xi|}\right) \int_0^{|\xi|} \exp\left(-\tfrac{1}{2}t^2\right) dt \right\}, \end{aligned}$$

and for $|\xi| = 0$

$$B(x) = 8\pi h^4.$$

With

$$\begin{aligned} \frac{\partial^2 B(x)}{\partial x_i \partial x_j} &= 4\pi h^2 \delta_{ij} \frac{1}{|\xi|^2} \left\{ \exp\left(-\tfrac{1}{2}|\xi|^2\right) + \left(|\xi| - \frac{1}{|\xi|}\right) \int_0^{|\xi|} \exp\left(-\tfrac{1}{2}t^2\right) dt \right\} \\ &+ 4\pi h^2 \frac{\xi_i \xi_j}{|\xi|^4} \left\{ -3 \exp\left(-\tfrac{1}{2}|\xi|^2\right) + \left(\frac{3}{|\xi|} - |\xi|\right) \int_0^{|\xi|} \exp\left(-\tfrac{1}{2}t^2\right) dt \right\} \end{aligned}$$

we obtain for $|\xi| \neq 0$

$$\begin{aligned} A_{ij}^{m,h}(x) &= \frac{1}{8\pi} \frac{1}{\sqrt{(2\pi)^3}} \left(2\delta_{ij} S(x) - \frac{\partial^2}{\partial x_i \partial x_j} B(x) \right) \\ &= \frac{1}{2\sqrt{(2\pi)^3}} \delta_{ij} h^2 \frac{1}{|\xi|^2} \left\{ -\exp\left(-\frac{|\xi|^2}{2}\right) + \left(|\xi| + \frac{1}{|\xi|}\right) \int_0^{|\xi|} \exp\left(-\frac{t^2}{2}\right) dt \right\} \\ &+ \frac{1}{2\sqrt{(2\pi)^3}} h^2 \frac{\xi_i \xi_j}{|\xi|^4} \left\{ 3 \exp\left(-\frac{|\xi|^2}{2}\right) + \left(|\xi| - \frac{3}{|\xi|}\right) \int_0^{|\xi|} \exp\left(-\frac{t^2}{2}\right) dt \right\}, \end{aligned}$$

and for $|\xi| = 0$ it follows finally

$$A_{ij}^{m,h}(x) = \frac{2}{3\sqrt{(2\pi)^3}}\delta_{ij}h^2.$$

□

5 Numerical calculations for the Poisson equation

In the following we present the results of some numerical simulations using the above formulas for the two-dimensional Poisson equation. Let us choose $3 \le \beta \in \mathbb{N}$ and define the test function

$$v(x_1, x_2) := \begin{cases} 16^\beta \left(\frac{1}{4} - x_1^2\right)^\beta \left(\frac{1}{4} - x_2^2\right)^\beta & \text{in} \quad Q, \\ 0 & \text{in} \quad \mathbb{R}^2 \backslash Q, \end{cases} \tag{16}$$

where

$$Q := \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, |x_1| < \tfrac{1}{2},\ |x_2| < \tfrac{1}{2} \right\}$$

denotes the open two-dimensional unit square. For $f := -\triangle v$ we obtain in $x \in Q$

$$\begin{aligned} f(x) = f(x_1, x_2) \;=\; & 2\,\beta\, 16^\beta \left(\tfrac{1}{4} - x_1^2\right)^{\beta-2} \left(\tfrac{1}{4} - x_2^2\right)^{\beta-2} \cdot \\ & \cdot \Bigg[\left(\tfrac{1}{4} - x_1^2\right)\left(\tfrac{1}{4} - x_2^2\right)\left\{\left(\tfrac{1}{4} - x_1^2\right) + \left(\tfrac{1}{4} - x_2^2\right)\right\} \\ & - 2(\beta - 1)\left\{ x_2^2 \left(\tfrac{1}{4} - x_1^2\right)^2 + x_1^2 \left(\tfrac{1}{4} - x_2^2\right)^2 \right\} \Bigg], \end{aligned} \tag{17}$$

and for $x \in \mathbb{R}^2 \backslash Q$ we have $f = 0$. Hence f is continuous in $\mathbb{R}^2$ if $\beta \ge 3$. The exponential integral function $\mathrm{exint}(x)$ in Theorem 2 has been evaluated with help of the NAG Fortran Library (see www.nag.co.uk).

6 Numerical calculations for the Stokes equations

For $4 \le \beta \in \mathbb{N}$ we define the test function $u = (u_1, u_2)$ with

$$\left.\begin{aligned} u_1(x_1, x_2) &= 4x_2 \left(\tfrac{16}{3}\right)^{2\beta-1} \left(\tfrac{1}{4} - x_1^2\right)^\beta \left(\tfrac{1}{4} - x_2^2\right)^{\beta-1} \\ u_2(x_1, x_2) &= -4x_1 \left(\tfrac{16}{3}\right)^{2\beta-1} \left(\tfrac{1}{4} - x_2^2\right)^\beta \left(\tfrac{1}{4} - x_1^2\right)^{\beta-1} \end{aligned}\right\} \text{ in } Q \tag{18}$$

The error $\varepsilon_h := \max|v(x) - v_h(x)|$ for different values of the smoothness parameter β is shown in Table 1.

h	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$
0,1	1,41487e-01	2,49694e-01	2,97605e-01	3,34063e-01
0,05	4,23751e-02	7,49503e-02	9,22317e-02	1,08464e-01
0,025	1,10846e-02	1,96726e-02	2,44883e-02	2,92301e-02
0,0125	2,80271e-03	4,97935e-03	6,21759e-03	7,45106e-03
0,00625	7,02666e-04	1,24871e-03	1,56047e-03	1,87193e-03
0,003125	1,75791e-04	3,12419e-04	3,90498e-04	4,68558e-04
0,0015625	4,39555e-05	7,81200e-05	9,76483e-05	1,17175e-04

Table 1. Maximal error expansion

The corresponding order $\alpha_h := \log_2 \frac{\varepsilon_{2h}}{\varepsilon_h}$ of convergence is presented in Table 2 and confirms an approximate approximation of second order.

h	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$
0,05	1,73938	1,73616	1,69007	1,62291
0,025	1,93466	1,92975	1,91317	1,89169
0,0125	1,98366	1,98216	1,97767	1,97194
0,00625	1,99591	1,99552	1,99438	1,99292
0,003125	1,99898	1,99888	1,99860	1,99823
0,0015625	1,99974	1,99972	1,99965	1,99956

Table 2. Order of convergence

and $u = 0$ in $\mathbb{R}^2 \setminus Q$, where Q is the unit square as in Section 5. An easy calculation shows $\operatorname{div} u = 0$ in $\mathbb{R}^2$.

Moreover, setting

$$p(x_1, x_2) := 16^{\beta-1} \left(\frac{1}{4} - x_1^2\right)^{\beta-1} \left(\frac{1}{4} - x_2^2\right)^{\beta-1} \quad \text{in } Q \tag{19}$$

and $p = 0$ in $\mathbb{R}^2 \backslash Q$, we obtain for the function $f := -\triangle u + \nabla p$ in $x \in Q$ the representation

$$\begin{aligned} f_i(x) = f_i(x_1,\, x_2) \;=\; & (-1)^i 8 \left(\frac{16}{3}\right)^{2\beta-1} x_{\tilde{i}} \left(\frac{1}{4} - x_i^2\right)^{\beta-2} \left(\frac{1}{4} - x_{\tilde{i}}^2\right)^{\beta-3} \cdot \\ & \cdot \left[(\beta-1) \left(\frac{1}{4} - x_i^2\right)^2 \left\{ 2(\beta-2) x_{\tilde{i}}^2 - 3 \left(\frac{1}{4} - x_{\tilde{i}}^2\right) \right\} \right. \\ & \left. + \beta \left(\frac{1}{4} - x_{\tilde{i}}^2\right)^2 \left\{ 2(\beta-1) x_i^2 - \left(\frac{1}{4} - x_i^2\right) \right\} \right] \\ & - 16^{\beta-1} 2(\beta-1) x_i \left(\frac{1}{4} - x_i^2\right)^{\beta-2} \left(\frac{1}{4} - x_{\tilde{i}}^2\right)^{\beta-1}, \end{aligned} \tag{20}$$

where $i = 1, 2$ and $\tilde{i} = \begin{cases} 2, & i = 1 \\ 1, & i = 2 \end{cases}$. In addition, for $x \in \mathbb{R}^2 \backslash Q$ we have $f_i = 0$.

The error $\varepsilon_h := \max |u_i(x) - u_h^i(x)|$ (the results are identical for $i = 1, 2$) for different values of the smoothness parameter β is shown in Table 3.

h	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$
0,1	1,08505e-00	2,32411e-00	4,22133e-00	7,34847e-00
0,05	3,47030e-01	7,68388e-01	1,44394e-00	2,60742e-00
0,025	9,25330e-02	2,07515e-01	3,95278e-01	7,23272e-01
0,0125	2,35129e-02	5,29152e-02	1,01176e-01	1,85807e-01
0,00625	5,90226e-03	1,32948e-02	2,54448e-02	4,67728e-02
0,003125	1,47707e-03	3,32784e-03	6,37069e-03	1,17134e-02
0,0015625	3,69362e-04	8,32221e-04	1,59327e-03	2,92961e-03

Table 3. Maximal error expansion

The corresponding order $\alpha_h := \log_2 \frac{\varepsilon_{2h}}{\varepsilon_h}$ of convergence is presented in Table 4 and confirms an approximate approximation of second order, too.

h	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$
0,05	1,64463	1,59677	1,54769	1,49482
0,025	1,90702	1,88862	1,86907	1,85001
0,0125	1,97651	1,97146	1,96600	1,96073
0,00625	1,99411	1,99282	1,99142	1,99007
0,003125	1,99853	1,99820	1,99785	1,99751
0,0015625	1,99963	1,99955	1,99946	1,99938

Table 4. Order of convergence

References

[1] W. Hackbusch, Integralgleichungen, Teubner, Stuttgart, 1989.

[2] T. Ivanov, Boundary Layer Approximate Approximations and Cubature of Potentials in Domains, PhD-thesis, Linköping University, 1997.

[3] V. Maz'ya, A New Approximation Method and its Applications to the Calculation of Volume Potentials. Boundary Point Method, In: 3. Kolloquium des DFG-Forschungsschwerpunktes Randelementmethoden, 1991.

[4] V. Maz'ya, Approximate Approximations, In: The Mathematics of Finite Elements and Applications, Highlights 1993, J. R. Whiteman (ed.), 77-104, Wiley, Chichester, 1994.

[5] V. Maz'ya, G. Schmidt, Approximate Approximations and the Cubature of Potentials, Rend. Lincei (9) Mat. Appl., 6 (1995), 161-184.

[6] V. Maz'ya, G. Schmidt, On Approximate Approximations using Gaussian Kernels, IMA J. Num. Anal., 16 (1969), 13-29.

[7] V. Maz'ya, G. Schmidt, Construction of Basis Functions for High Order Approximate Approximations, In: Mathematical Aspects of Boundary Element Methods, M. Bonnet et al. (eds.), Chapman Hall/CRC Res. Notes Math., 414 (2000), 191-202.

[8] V. Maz'ya, G. Schmidt and W. Wendland, On the Computation of Multidimensional Layer Harmonic Potentials via Approximate Approximations, Calc., 40 (2003), 33-53.

[9] F. Müller, W. Varnhorn, An Approximation Method using Approximate Approximations, Applic. Anal., 85 (2006), 669-680.

[10] F. Müller, W. Varnhorn, Error Estimates for Approximate Approximations on Compact Intervals, J. A. T., 145 (2007), 171-181.

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