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Introduction of Fréchet and Gâteaux Derivative

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Abstract

In this paper the Fréchet and Gâteaux differentiation of functions on Banach space has been introduced. We investigate the algebraic properties of Fréchet and Gâteaux differentiation and its relation by giving some examples.

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1 Introduction

In mathematics, the Fréchet derivative is a derivative define on Banach spaces. Gâteaux derivative is a generalization of the concept of directional derivative in differential calculus.
In this reviwe we contrast Fréchet derivative to the more general Gâteaux derivative. Both derivative are often used to formalize the functional derivative commonly used in Physics, particulary Quantum field theory. The purpose of this work is to review some results obtained on the structure of the derivative of Fréchet and Gâteaux.

2 Recall

Definition 2.1 *suppose* $f : U \subseteq R^n \to R^m$ *where* U *is an open set. The function* f *is classically differentiable at* $x_0 \in U$ *if*

(i)The partial derivatives of f, $\frac{\partial f_i}{\partial x_j}$ *for* $i = 1, ..., m$ *and* $j = 1, ..., n$ *exists at* x_0.

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(ii)The Jacobian matrix $J(x_0) = [\frac{\partial f_i}{\partial x_j}(x_0)] \in R^{m\times n}$ *satisfies*

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - J(x_0)(x - x_0)\|}{\|x - x_0\|} = 0.$$

we say that the Jacobian matrix $J(x_0)$ *is the derivative of* f *at* x_0*, that is called total derivative.*

Definition 2.2 *Let* X, Y *are Banach spaces, the directional derivative of* $f : X \to Y$ *at* $x \in U \subseteq X$ *in the direction* $h \in X$*, denoted by the symbol* $f'(x; h)$*, is defined by the equation*

$$f'(x; h) = \lim_{t\to 0} \frac{f(x + th) - f(x)}{t},$$

whenever the limit on the right exists.

3 Fréchet and Gâteaux derivative

Definition 3.1 *let* f *be a function on an open subset* U *of a Banach space* X *into the Banach space* Y*. we say* f *is Gâteaux differentiable at* $x \in U$ *if there is bounded and linear operator* $T : X \to Y$ *such that*

$$\lim_{t\to 0} \frac{f(x + th) - f(x)}{t} = T_x(h)$$

for every $h \in X$*. The operator* T *is called the Gâteaux derivative of* f *at* x*.*

Definition 3.2 *let* f *be a function on an open subset* U *of a Banach space* X *into the Banach space* Y*. we say* f *is Fréchet differentiable at* $x \in U$ *if there is bounded and linear operator* $T : X \to Y$ *such that*

$$\lim_{t\to 0} \frac{f(x + th) - f(x)}{t} = T_x(h)$$

is uniform for every $h \in S_X$*. The operator* T *is called the Fréchet derivative of* f *at* x*.*

Remark: if we set $th = y$ then $t \to 0$ iff $y \to o$. therefore $f : X \to Y$ is Fréchet differentiable at $x \in U$ if

$$\lim_{y\to o} \frac{f(x + y) - f(x) - T(y)}{\|y\|} = 0$$

for all $y \in X$.

Definition 3.3 *Let f be a real-valued function on an open subset U of a Banach space X. we say that f is uniformly Gâteaux differentiable on U if for every $h \in S_X$*

$$\lim_{t\to 0}\frac{f(x+th)-f(x)}{t}=T_x(h)$$

is uniform in $x \in U$.

Definition 3.4 *Let f be a real-valued function on an open subset U of a Banach space X. we say that f is uniformly Fréchet differentiable on U if*

$$\lim_{t\to 0}\frac{f(x+th)-f(x)}{t}=T_x(h)$$

is uniform in $h \in S_X$ and $x \in U$.

4 Theorems and properties

Theorem 4.1 *[3] let f be a convex function defined on an open convex subset U of a Banach space X that is continuous at $x \in U$. then f is fréchet differentiable at x iff*

$$\lim_{t\to 0}\frac{f(x+th)+f(x-th)-2f(x)}{t}=0$$

uniformly for $h \in S_X$.

we say that a mean value theorem can be derived for Gâteaux or Fréchet differentiable functions.

Lemma 4.2 *let X and Y be Banach spaces, let U an open subset of X, and let a mapping*

$$F : U \to Y$$

be Gâteaux differentiable at every point of the interval

$$[x, x+h] \subset U.$$

then,

$$\|F(x+h)-F(x)\| \le \sup_{0\le t\le 1}\|F'(x+th)\|\,\|h\|\,.$$

Obviously Fréchet differentiability has additive property and the product of two Fréchet differentiable functions is Fréchet differentiable function.(the function that is Fréchet differentiable is continuous and therefore locally bounded, now we use boundedness Fréchet derivative and triangle inequality.)
A function which is Fréchet differentiable at a point, is continuous there. this is not the case for Gâteaux differentiable functions even in finite dimensions.

Example 4.3 *the function $f : R^2 \to R^2$ defined by*

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{x^4 y}{x^6+y^3} & x^2+y^2>0, \end{cases}$$

has 0 as its Gâteaux derivative at the origin but fails to be continuous there.

Remark: The product of two Gâteaux differentiable functions is not Gâteaux differentiable necessary .

Lemma 4.4 *[2] Let X and Y be Banach spaces. let $f_n : X \to Y$ be Gâteaux differentiable mapping, for all n. Assume that $(\sum f_n)$ converge pointwise on X, and that there exists a constant $K > 0$ so that for all h,*

$$\sum_{n\geq 1} \sup_{x\in X} \left\| \frac{\partial f_n}{\partial h}(x) \right\| \leq K \|h\| .$$

Then the mapping $f = \sum\limits_{n\geq 1} f_n$ is Gâteaux differentiable on X for all x and $f'(x) = \sum\limits_{n\geq 1} f'_n(x)$.

Lemma 4.5 *suppose that X, Y and Z are Banach spaces and $g : X \to Y$ and $f : Y \to Z$ are Fréchet differentiable.The derivative of the composition $f \circ g$ is given by the chain rule*

$$(f \circ g)'(x) = f'(g(x))(g'(x)).$$

The theorem on differentiation of a composite function is usually invalid for the Gâteaux derivative.

Example 4.6 *Let $f : R \to R^2$ and $f(x) = (x, x^2)$ and $g : R^2 \to R$ with*

$$g(x,y) = \begin{cases} x & x = y^2 \\ 0 & o.w. \end{cases}$$

are define. we have $g \circ f = 1$ but $g'(0) = 0$ and $f'(0) = (h, o)$. then we didn't have chain rule.

5 Main Result

Every uniformly Fréchet differentiable function is a function which is uniformly Gâteaux differentiable, Fréchet differentiable and Gâteaux differentiable, but the converse is not true necessary.
It is obvious that f is uniformly Fréchet differentiable on an open convex set U iff it is Fréchet differentiable at every point of U and the map $x \mapsto f'(x)$ is uniformly continuous as a map $U \to X^*$.

Example 5.1 *[4] For u in c_0 enumerate the support of u as $\{n_k\}$ such that $|u(n_k)| \geq |u(n_{k+1})|$ for $k = 1, 2, ...,$*
define Du in ℓ^2 by

$$Du(n) = \begin{cases} \frac{u(n_k)}{2^k} & if n = n_k for some k \\ 0 & o.w \end{cases}$$

and deine $|||u||| = \|Du\|_2$.
For $x = (x^1, x^2, ...)$ in ℓ^2 let

$$u = (\frac{1}{2}\|x\|_2, x^1, x^2, x^2, ..., \underbrace{x^j, x^j, ..., x^j,}_{j} ...)$$

be the element of c_0 associated with x and define

$$\|x\|_L = |||u|||.$$

then $\|x\|_L$ in $(\ell^2, \|.\|_L)^*$ is Fréchet differentiable but not uniformly Gâteaux differentiable.

Example 5.2 *[4] For $x = (x^1, x^2, ...)$ in ℓ^2 let $x' = (0, x^2, ...)$ and define the equivalent norm $\|x\|_S = \max\{|x^1|, \|x'\|_2|\}$. Let $\{\alpha_n\}$ be a sequence of positive real numbers decreasing to zero and define the continuous linear injection*
$T : \ell^2 \to \ell^2$ by $T(x^1, x^2, ...) = (x^1, \alpha_2 x^2, \alpha_3 x^3, ...)$. For x in ℓ^2 define

$$\|x\|_W = (\|x\|_S^2 + \|Tx\|_2^2)^{\frac{1}{2}}.$$

Then $\|x\|_W$ in $(\ell^2, \|.\|_W)^*$is uniformly Gâteaux differentiable but not Fréchet differentiable.

Every Fréchet differentiable function is Gâteaux differentiable, but the converse is not true.

Example 5.3 *Canonical norm of ℓ_1 is nowhere Fréchet differentiable and is Gâteaux differentiable at $x = (x_i)$ iff $x_i \neq 0$ for every i.*

Let X and Y be Banach spaces. $f : X \to Y$ is Lipschitz and $\dim(X) < \infty$ then Gâteaux differentiability and Fréchet differentiability coincide.
(Sketch of proof: using the compactness of the unit ball and the Lipschitz property of f.)
with the mean value theorem, we can derive useful conditions that guarantee that a given Gâteaux differentiable function is in fact Fréchet differentiable.

Indeed, let X, Y are Banach spaces and $f : X \to Y$ be a continuous mapping on $U \subset X$ at $x_0 \in U$. Assume that the mapping is Gâteaux differentiable at every point of U, and that the mapping $x \mapsto f'(x)$ from U into $L(X, Y)$ is continuous. then f is Fréchet differentiable on U for all $x \in U$.

If a Lipschitz function on a Banach space is differentiable at x_0 in all directions, then it is not necessary Gâteaux differentiable. for example in

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0), \end{cases}$$

the directional derivative need not be linear. Thus f is Gâteaux differentiable at x iff all the directional derivative $f'(x;h)$ exists and they form a bounded linear operator of h. therefore we have in this case $f'(x,h) = T_x(h)$.

References

[1] Y. Benyamini, J. Lindenstrauss, Geometric nonlinear functional Analysis, vol.1, American Mathematical society providence, Rholde Island,(1999).

[2] R. Deville, P. Hajek, On The Range Of The Derivative Of The Gâteaux-smooth functions On Separble Banach space, Israel J. Math, 145 (2005), 257-269.

[3] M. Fabian, P. Habala, P. Hajek, V. M. Santalucia, J. Pelant, V. Zizler, Functioanal Analysis Infinite-dimensuional Geometry (2001).

[4] M. A. Smith, Some example concerning Rotundity in Banach Spaces, Math. Ann., 233 (1978), 155-161.

[5] S. Yamamuro, Differential Calauls in Topologycal Linear Spaces, Lecture Notes in Mathemayics, 374, Springer (1970).

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