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Conditioned Limit Theorems for Weighted Sums of Random Sequence[1]

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Abstract. Some notions of conditionally dominated random variables are introduced and characterized, Under rather minimal assumptions on random variables $\{X, X_n, n \geq 1\}$, some limit theorems of Jamison's type of weighted sums of random variables are obtained.

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1. Introduction

It is of interesting in probability theory and statistics to consider the convergence of weighted sums $\sum_{k=1}^{n} w_{nk}[X_k - E(X_k|\mathcal{F}_{k-1})]$, see e.g. [2],[4],[5],[6] and many results have been made in the field. Conditions of independent random variables are basic in historic results due to Bernoulli, Borel and Komogrov(cf.[5]). Recently, serious attempts have been made to relax these strong conditions(cf.[4]). Such as stochastically dominated conditions of some kind, and these have played an increasingly important role as a key condition in proving laws of large numbers. In Y. Adler and A. Rosasky, for example, the authors considered a sequence of independent random variables.

In the present paper, we are interesting in introducing a new set of conditions to be called conditionally dominated in Cesàro sense concerning the array $\{w_{nk}\}$ for a sequence of random variables, and we will show some stability results of Jamison type weighted sums of arbitrary random variables in

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more general settings.

2. Preliminary work

Some definitions and preliminary results will be presented prior to establish the main results.

Let $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu, n \in \mathbb{N})$ be a probability space and $\mathbb{N}$ denote the set of non-negativ integer, where $\{\mathcal{F}_n = \sigma(X_0, \cdots, X_n), n \in \mathbb{N}\}$ is an increasing sequence of sub $\sigma-$ algebras of the basic $\sigma-$algebra $\mathcal{F}$ and $\mathcal{F}_0$ is the trivial $\sigma-$algebra $\{\phi, \Omega\}$. Suppose that $\{X, X_n, n \in \mathbb{N}\}$ be a stochastic sequence defined on this underlying probability space.

For all $n \in \mathbb{N}$, $A, A_0, A_1, \cdots, A_n \in \mathcal{B}$($\mathcal{B}$ is the Borel σ algebra on $\mathbb{R}$),

$$\mu\{X \in A\} = \int_{x \in A} p(x)\mu(dx) \tag{1}$$

and

$$\mu\{X_0 \in A_0, \cdots, X_n \in A_n\} = \int_{x_0 \in A_0} \cdots \int_{x_n \in A_n} p_n(x_0, \cdots, x_n)\mu(dx_0 \cdots dx_n) \tag{2}$$

and denote the conditional pmf(pdf) by

$$p_n(x_n|x_0, \cdots, x_{n-1}) = \frac{p_n(x_0, \cdots, x_n)}{p_{n-1}(x_0, \cdots, x_{n-1})} \tag{3}$$

where $p(x), p_n(\cdots)$ are probability mass functions(pmf) or probability density functions(pdf) w.r.t.μ. In nearly all cases μ and is either the Lebesgue or counting measures.

Let $\{a_k, k \geq 1\}$ be a sequence of positive real numbers, $b = \sup\{a_k, k \geq 1\} < \infty$,$\mathbf{W} = \{w_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of positive real numbers, where

$$w_{nk} = \begin{cases} a_k/\sigma_n, & for\ k \leq n \\ 0, & for\ k > n \end{cases}$$

satisfying $\sum_{k=1}^n w_{nk} \leq 1$ and with $\sigma_n \uparrow \infty$. We shall study the Jamison type weighted sums of the form

$$\mathbb{T}_n(\mathbf{W}) = \sum_{k=1}^n w_{nk}[X_k - \mathbb{E}(X_k|\mathcal{F}_{k-1})] \tag{4}$$

for all $n \in \mathbb{N}$.

Definition 1. Let$\{ X, X_n, n \in \mathbb{N}\}$be a sequence of random variables, the conditionally moment generating function and conditional tail probability moment generating function of X_k with respect to a_k as follows:

$$\mathbb{M}_n(s; x_0, \cdots, x_{n-1}) = \int_{-\infty}^{\infty} e^{a_n x s} p_n(x|x_0, \cdots, x_{n-1})\mu(dx). \tag{5}$$

$$\begin{aligned} & \widetilde{\mathbb{M}}_n(s; x_0, \cdots, x_{n-1}) \\ = & \int_0^\infty e^{a_n x s} \int_x^\infty p_n(t|x_0, \cdots, x_{n-1}) dt \mu(dx) \\ - & \int_{-\infty}^0 e^{a_n x s} \int_{-\infty}^x p_n(t|x_0, \cdots, x_{n-1}) dt \mu(dx). \end{aligned} \tag{6}$$

and let

$$\widetilde{\mathbb{M}}_(s) = \int_0^\infty e^{bxs} \int_x^\infty p(t) dt \mu(dx) - \int_{-\infty}^0 e^{bxs} \int_{-\infty}^x p(t) dt \mu(dx). \tag{7}$$

$$\widetilde{\mathbb{M}}^+(s) = \int_0^\infty e^{bxs} \int_x^\infty p(t) dt \mu(dx), \quad \widetilde{\mathbb{M}}^-(s) = \int_{-\infty}^0 e^{bxs} \int_{-\infty}^x p(t) dt \mu(dx). \tag{8}$$

provided that the integrals exit for $s \in (-s_0, s_0)$ for some $s_0 > 0$.

Definition 2. *(cf.[6])Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of random variables, and is said to be :*

1) conditionally dominated by a random variable X in double sides(we write $\{X_n, n \in \mathbb{N}\} \prec X$) if there exists a constant $C > 0$, for almost every $\omega \in \Omega$, such that

$$\sup_{n \in \mathbb{N}} \mu\{X_n > x | \mathcal{F}_{n-1}\} \le C \mu\{X > x\} \quad for\ all\ x > 0.$$

and

$$\sup_{n \in \mathbb{N}} \mu\{X_n < x | \mathcal{F}_{n-1}\} \le C \mu\{X < x\} \quad for\ all\ x < 0.$$

2) conditionally dominated in Cesàro sense by a random variable X, concerning the array $\{w_{nk}\}$, in double sides(we write $\{X_n, n \in \mathbb{N}\} \prec X(C)$) if there exists a constant $C > 0$, for almost every $\omega \in \Omega$, such that

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^n w_{nk} \mu\{X_k > x | \mathcal{F}_{k-1}\} \le C \mu\{X > x\} \quad for\ all\ x > 0.$$

and

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^n w_{nk} \mu\{X_k < x | \mathcal{F}_{k-1}\} \le C \mu\{X < x\} \quad for\ all\ x < 0.$$

Remark. In the particular case of array

$$w_{nk} = \begin{cases} 1/n, & for\ k \le n \\ 0, & for\ k > n \end{cases}$$

the condition of $\{w_{nk}\}$-stochastically dominated in Cesàro sense is the dominated in Cesàro sense.

Lemma 1. *Let $\mathbb{M}_n(s; x_0, \cdots, x_{n-1}), \widetilde{\mathbb{M}}_n(s; x_0, \cdots, x_{n-1})$ be defined as above, then*

$$\frac{\mathbb{M}_n(s; x_0, \cdots, x_{n-1}) - 1}{s} = a_n \widetilde{\mathbb{M}}_n(s; x_0, \cdots, x_{n-1}) \tag{9}$$

and

$$\widetilde{\mathbb{M}}_n(0) = \mathbb{E}(X_n|x_0,\cdots,x_{n-1}) \tag{10}$$

Proof. Since

$$\frac{\mathbb{M}_n(s;x_0,\cdots,x_{n-1})-1}{s} = \int_0^\infty \frac{e^{a_nsx}-1}{s} p_n(x|x_0,\cdots,x_{n-1})\mu(dx)$$

$$+\int_{-\infty}^0 \frac{e^{a_nsx}-1}{s} p_n(x|x_0,\cdots,x_{n-1})\mu(dx)$$

$$= \int_0^\infty \frac{1-e^{a_nsx}}{s} d\int_x^\infty p_n(t|x_0,\cdots,x_{n-1})dt$$

$$+\int_{-\infty}^0 \frac{1-e^{a_nsx}}{s} d\int_{-\infty}^x p_n(t|x_0,\cdots,x_{n-1})dt$$

$$= [\frac{1-e^{a_nsx}}{s}\int_x^\infty p_n(t|x_0,\cdots,x_{n-1})dt]|_0^\infty$$

$$+a_n\int_0^\infty e^{a_nsx}\int_x^\infty p_n(t|x_0,\cdots,x_{n-1})dt\mu(dx)$$

$$+[\frac{1-e^{a_nsx}}{s}\int_{-\infty}^x p_n(t|x_0,\cdots,x_{n-1})dt]|_0^\infty$$

$$-a_n\int_{-\infty} e^{a_nsx}\int_{-\infty}^x p_n(t|x_0,\cdots,x_{n-1})dt\mu(dx) = a_n\widetilde{\mathbb{M}}_n(s;x_0,\cdots,x_{n-1}).$$

(9) follows. (10) can also be obtained immediately from integration by parts. □

3. Mainstream

Theorem 1. *Let $\{X, X_n, n\in\mathbb{N}\}$ be defined as before. If $\{X_n, n\in\mathbb{N}\} \prec X$ with $\mathbb{E}X<\infty$ and let $\sigma_n \uparrow \infty$ as $n\to\infty$. Then*

$$\lim_n \mathbb{T}_n(\mathbf{W}) = 0, \quad \mu - a.s. \tag{11}$$

Proof. Putting

$$p_k(s;x_k) = \frac{e^{a_kx_ks}p_k(x_k|x_0,\cdots,x_{k-1})}{\mathbb{M}_k(s)}$$

and

$$\widetilde{p}_n(s;x_0,\cdots,x_n) = p_0(x_0)\prod_{k=1}^n p_k(s;x_k), \quad n=1,2,\cdots.$$

Therefore $\widetilde{p}_n(s;x_0,\cdots,x_n)$ is a pmf or pdf of $n+1$ variables, let us define

$$\Lambda_n(s,\omega) = \begin{cases} \frac{\widetilde{p}_n(s;X_0,\cdots,X_n)}{p_n(X_0,\cdots,X_n)}, & if\ the\ denominator > 0 \\ 0, & otherwise \end{cases}$$

From reference [3], we have

$$\limsup_n \sigma_n^{-1} \log \Lambda_n(s;\omega) \le 0, \qquad \mu - a.s. \tag{12}$$

Note that

$$\log \Lambda_n(s;\omega) = s\sum_{k=1}^n a_k X_k - \sum_{k=1}^n \log \mathbb{M}_k(s; X_0, \cdots, X_{k-1}) \tag{13}$$

By (12) and (13),we have

$$\limsup_n \frac{1}{\sigma_n}[s\sum_{k=1}^n a_k X_k - \sum_{k=1}^n \log \mathbb{M}_k(s; X_0, \cdots, X_{k-1})] \le 0,\ \mu - a.s. \tag{14}$$

Thus

$$\limsup_n s\sum_{k=1}^n w_{nk} X_k \le \limsup_n \frac{1}{\sigma_n}\sum_{k=1}^n \log \mathbb{M}_k(s; X_0, \cdots, X_{k-1}),\ \mu - a.s. \tag{15}$$

By the property of the superior limit $\limsup_n(a_n - b_n) \le 0 \to \limsup_n(a_n - c_n) \le \limsup_n(b_n - c_n)$ and note that $\mathbb{E}(X_k|\mathcal{F}_{k-1}) < \infty,\ a.s.\ k = 1, 2, \cdots$. Dividing two sides of (15) by s, we have, by lemma 1, for any $s \in (-s_0, 0)$,

$$\liminf_n \sum_{k=1}^n w_{nk}[X_k - \mathbb{E}(X_k|\mathcal{F}_{k-1})]$$

$$\ge \liminf_n \frac{1}{\sigma_n}\sum_{k=1}^n [\frac{\log \mathbb{M}_k(s; X_0, \cdots, X_{k-1})}{s} - a_k \mathbb{E}(X_k|\mathcal{F}_{k-1})],\ \mu - a.s. \tag{16}$$

From the inequality $\log x \le x - 1 (x > 0)$ and lemma 1, we have

$$\liminf_n \mathbb{T}_n(\mathbf{W}) \ge \liminf_n \frac{1}{\sigma_n}\sum_{k=1}^n [\frac{\mathbb{M}_k(s; X_0, \cdots, X_{k-1}) - 1}{s} - a_k \mathbb{E}(X_k|\mathcal{F}_{k-1})]$$

$$= \liminf_n \frac{1}{\sigma_n}\sum_{k=1}^n a_k[\widetilde{\mathbb{M}}_k(s; X_0, \cdots, X_{k-1}) - \mathbb{E}(X_k|\mathcal{F}_{k-1})],\ \mu - a.s. \tag{17}$$

Let

$$\varphi(s) = \liminf_n \sum_{k=1}^n w_{nk}[\widetilde{\mathbb{M}}_k(s; X_0, \cdots, X_{k-1}) - \mathbb{E}(X_k|\mathcal{F}_{k-1})], s \in (-s_0, 0) \tag{18}$$

If $-s_0 \le s < s + \Delta s < 0$, by (18) and noticing that $\{X_n, n \in \mathbb{N}\} \prec X$ and $\sum_{k=1}^n w_{nk} \le 1$, we have

$$\varphi(s + \Delta s) - \varphi(s)$$

$$= \liminf_n \sum_{k=1}^n w_{nk}[\widetilde{\mathbb{M}}_k(s + \Delta s; X_0, \cdots, X_{k-1}) - \widetilde{\mathbb{M}}_k(0; X_0, \cdots, X_{k-1})]$$

$$-\liminf_{n}\sum_{k=1}^{n}w_{nk}[\widetilde{\mathbb{M}}_k(s;X_0,\cdots,X_{k-1})-\widetilde{\mathbb{M}}_k(0;X_0,\cdots,X_{k-1})]$$

$$=\liminf_{n}\sum_{k=1}^{n}w_{nk}[\widetilde{\mathbb{M}}_k(s+\Delta s;X_0,\cdots,X_{k-1})-\widetilde{\mathbb{M}}_k(0;X_0,\cdots,X_{k-1})]$$

$$+\limsup_{n}\sum_{k=1}^{n}w_{nk}[\widetilde{\mathbb{M}}_k(0;X_0,\cdots,X_{k-1})-\widetilde{\mathbb{M}}_k(s;X_0,\cdots,X_{k-1})]$$

$$\leq\limsup_{n}\sum_{k=1}^{n}w_{nk}[\widetilde{\mathbb{M}}_k(s+\Delta s;X_0,\cdots,X_{k-1})-\widetilde{\mathbb{M}}_k(s;X_0,\cdots,X_{k-1})]$$

$$=\limsup_{n}\sum_{k=1}^{n}w_{nk}[\int_0^{\infty}(e^{a_kt(s+\Delta s)}-e^{a_kts})\mu(X_k>t|\mathcal{F}_{k-1})dt$$

$$-\int_{-\infty}^{0}(e^{a_kt(s+\Delta s)}-e^{a_kts})\mu(X_k<t|\mathcal{F}_{k-1})dt]$$

$$=\limsup_{n}\sum_{k=1}^{n}w_{nk}[\int_0^{\infty}e^{a_kts}(e^{a_kt\cdot\Delta s}-1)\mu(X_k>t|\mathcal{F}_{k-1})dt$$

$$-\int_{-\infty}^{0}e^{a_kts}(e^{a_kt\cdot\Delta s}-1)\mu(X_k<t|\mathcal{F}_{k-1})dt]$$

$$\leq C\limsup_{n}\sum_{k=1}^{n}w_{nk}[\int_0^{\infty}e^{a_kts}(e^{a_kt\cdot\Delta s}-1)\mu(X>t)dt$$

$$-\int_{-\infty}^{0}e^{a_kts}(e^{a_kt\cdot\Delta s}-1)\mu(X<t)dt]$$

$$\leq C\limsup_{n}\sum_{k=1}^{n}w_{nk}[\int_0^{\infty}(e^{bt\cdot\Delta s}-1)\mu(X>t)dt$$

$$-\int_{-\infty}^{0}e^{bts}(e^{bt\cdot\Delta s}-1)\mu(X<t)dt]$$

$$\leq C[\int_0^{\infty}(e^{bt\cdot\Delta s}-1)\mu(X>t)dt-\int_{-\infty}^{0}e^{bts}(e^{bt\cdot\Delta s}-1)\mu(X<t)dt]$$

$$=C\{[\widetilde{\mathbb{M}}^{+}(\triangle s)-\widetilde{\mathbb{M}}^{+}(0)]-[\widetilde{\mathbb{M}}^{-}(s+\triangle s)-\widetilde{\mathbb{M}}^{-}(s)]\}$$

which follows that $\varphi(s)$ is continuous on $(-s_0,0)$, let $s\to 0$ in (17), we obtain

$$\liminf_{n}\mathbb{T}_n(\mathbf{W})\geq 0,\ \ \mu-a.s.$$

Similarly, we can get

$$\limsup_{n}\mathbb{T}_n(\mathbf{A})\leq 0,\ \ \mu-a.s.$$

These complete the proofs of the Theorem 1. □

Theorem 2. *Let $\{X, X_n, n \in \mathbb{N}\}$ be defined as above and $\{X_n, n \in \mathbb{N}\} \prec X(C)$ with $\mathbb{E}X < \infty$. If $\sigma_n \uparrow \infty$ as $n \to \infty$, then*

$$\lim_n \mathbb{T}_n(\mathbf{W}) = 0, \quad \mu - a.s.$$

Corollary 1. *(SLLN) Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of independent random variables, if*

$$\sup_{n\in\mathbb{N}} \mu\{|X_n| > x\} \leq C\mu\{|X| > x\} \quad for\ all\ x > 0.$$

or

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} w_{nk}\mu\{|X_k| > x\} \leq C\mu\{|X| > x\} \quad for\ all\ x > 0.$$

then

$$\lim_n \mathbb{T}_n(\mathbf{W}) = 0, \quad \mu - a.s.$$

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