

Oscillation Criteria for a Class of Partial Difference Equations¹

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Abstract

In this paper we consider the partial difference equation with continuous variables, Some sufficient conditions for all solutions of this equation to be oscillatory are obtained.

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1 Introduction

Partial difference equations arise in applications involving population dynamics with spatial migrations, chemical reactions (see [1,2]). Recently, the qualitative analysis of Partial difference equations has received much more attention (see [3]). In particular, the oscillation of partial difference equations with continuous variables has been investigated in some papers (see [4-8] and the references therein). To further the qualitative analysis of Partial difference equations, in

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this paper we shall consider the Partial difference equation with continuous variables

$$\begin{aligned} d_1 A(x+a, y+b) + d_2 A(x+a, y) + d_3 A(x, y+b) - d_4 A(x, y) \\ + \sum_{i=1}^n p_i(x, y) A(x - \tau_i, y - \sigma_i) = 0 \end{aligned} \quad (1)$$

Where $p_i \in C(R^+ \times R^+, R^+)$, a, b, τ_i and σ_i are positive, $d_1 \geq 0, d_2, d_3 \geq d_4 > 0$.

By a solution of (1), we mean a continuous function $A(x, y)$ which satisfies (1) for $x \geq x_0 \geq 0, y \geq y_0 \geq 0$. A solution $A(x, y)$ of equations is said to be eventually positive if $A(x, y) > 0$ for all large x and y , and eventually negative if $A(x, y) < 0$ for all large x and y . It is said to be oscillatory if it is neither eventually positive nor eventually negative.

In this paper we will concern with the oscillation of the equations (1). Equation (1) has been investigated by [8,9]. Our purpose is to obtain new oscillation criteria for the oscillation of (1) by new techniques. Our studies are motivated by the work of [10]. Some new sufficient conditions for this equation to be oscillatory are derived, which is easier to be verified than that of [9], and our results certainly extend/implement the oscillation results in [9].

2 Main results

In what follows we will assume the following for $i = 1, 2, \dots, n$:

(1) $\tau_i = k_i a + \theta_i, \quad \sigma_i = l_i b + \psi_i,$ where k_i, l_i are nonnegative integers, $\theta_i \in [0, a), \quad \psi_i \in [0, b)$;

(2)

$$Q_i(x, y) = \min\{p_i(z, w) | x \leq z \leq x+a, y \leq w \leq y+b, x \geq x_0 \geq 0, y \geq y_0 \geq 0\}$$

and

$$\liminf_{x, y \rightarrow \infty} Q_i(x, y) = q_i \geq 0, \quad i = 1, 2, \dots, n.$$

(3) $k_0 = \min\{k_i\}, \quad l_0 = \min\{l_i\}, \quad \eta_i = \min\{k_i, l_i\}, \quad i = 1, 2, \dots, n.$

Similar to [8,9], we have the following result:

Lemma 2.1 Let $A(x, y)$ be an eventually positive solution of equation (1).

Set

$$\omega(x, y) = \int_x^{x+a} \int_y^{y+a} A(u, v) du dv,$$

Then $\omega(x, y)$ is an eventually positive solution of the difference inequality

$$\begin{aligned} d_1 \omega(x+a, y+b) + d_2 \omega(x+a, y) + d_3 \omega(x, y+b) - d_4 \omega(x, y) \\ + \sum_{i=1}^n Q_i(x, y) \omega(x - k_i a, y - l_i b) \leq 0, \end{aligned} \quad (3)$$

and $\frac{\partial \omega}{\partial x} < 0$, $\frac{\partial \omega}{\partial y} < 0$.

In view of lemma 2.1, we have, for all x and y sufficiently large

Lemma 2.2

$$\begin{aligned} \omega(x+a, y+b) &< \omega(x, y), & \omega(x+a, y) &< \omega(x, y), \\ \omega(x, y+b) &< \omega(x, y), & \omega(x, y) &< \omega(x-k_i a, y-l_i b). \end{aligned}$$

The followings are the main results of the paper

Theorem 2.2 Suppose that one of the following four conditions holds :

(i) $k_0 > 0$, $l_0 > 0$:

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_1 + d_2 + d_3}{d_4} \right)^{\eta_i} \frac{(\eta_i + 1)^{\eta_i + 1}}{\eta_i^{\eta_i}} > 1; \quad (6)$$

(ii) $k_0 > 0$, $l_0 = 0$:

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_2}{d_4} \right)^{k_i} \frac{(k_i + 1)^{k_i + 1}}{k_i^{k_i}} > 1; \quad (7)$$

(iii) $k_0 = 0$, $l_0 > 0$:

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_3}{d_4} \right)^{l_i} \frac{(l_i + 1)^{l_i + 1}}{l_i^{l_i}} > 1; \quad (8)$$

(iv) $k_0 = 0$, $l_0 = 0$:

$$\frac{1}{d_4} \sum_{i=1}^n q_i > 1; \quad (9)$$

then every solution of equation (1) is oscillatory.

Proof: Suppose, to the contrary, $A(x, y)$ is an eventually positive solution of (1). Let $\omega(x, y)$ be defined as in lemma 2.1. We consider the above four cases:

Case 1 $k_0 > 0$, $l_0 > 0$: By the monotonicity of $\omega(x, y)$ with respect to x , y and Lemma 2.1, for sufficiently large x and y , we obtain

$$\begin{aligned} & \frac{(d_1 + d_2 + d_3)\omega(x+a, y+b)}{d_4\omega(x, y)} - 1 \\ & < \frac{d_1\omega(x+a, y+b) + d_2\omega(x+a, y) + d_3\omega(x, y+b)}{d_4\omega(x, y)} - 1 \\ & \leq -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \frac{\omega(x-k_i a, y-l_i b)}{\omega(x, y)} \\ & \leq -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \frac{\omega(x-\eta_i a, y-\eta_i b)}{\omega(x, y)} \\ & = -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \frac{\omega(x-\eta_i a, y-\eta_i b)}{\omega(x-\eta_i a+a, y-\eta_i b+b)} \dots \frac{\omega(x-a, y-b)}{\omega(x, y)}. \end{aligned} \quad (10)$$

Define

$$S(x, y) = \frac{\omega(x, y)}{\omega(x + a, y + b)}, \quad (11)$$

Notice that $\omega(x, y)$ is monotone decreasing and bounded, it is easy to see that $S(x, y)$ is bounded and also $S(x, y) > 1$. Substituting (11) into (10), we get

$$\frac{d_1 + d_2 + d_3}{d_4 S(x, y)} - 1 < -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \prod_{j=1}^{\eta_i} S(x - ja, y - jb). \quad (12)$$

Denote that $\xi = \lim_{x, y \rightarrow \infty} \inf S(x, y)$, Obviously $\xi \in [1, +\infty)$. From (12) and the definition of ξ , we have

$$\frac{d_1 + d_2 + d_3}{d_4 \xi} - 1 \leq -\frac{1}{d_4} \sum_{i=1}^n q_i \xi^{\eta_i}. \quad (13)$$

which implies that

$$\xi > \frac{d_1 + d_2 + d_3}{d_4}, \quad \frac{\sum_{i=1}^n q_i \xi^{\eta_i + 1}}{d_4 \xi - (d_1 + d_2 + d_3)} \leq 1 \quad (14)$$

Let

$$f(\xi) = \frac{\xi^{\eta_i + 1}}{d_4 \xi - (d_1 + d_2 + d_3)}, \quad (15)$$

Differentiate $f(\xi)$ with respect to ξ , we have

$$f'(\xi) = \frac{\xi^{\eta_i} [d_4 \xi \eta_i - (\eta_i + 1)(d_1 + d_2 + d_3)]}{[d_4 \xi - (d_1 + d_2 + d_3)]^2}$$

Let $f'(\xi) = 0$, we can get

$$\xi = \frac{d_1 + d_2 + d_3}{d_4} \cdot \frac{\eta_i + 1}{\eta_i}$$

Since $\eta_i \geq 0$, then we have

$$\min\{f(\xi)\} = \frac{1}{d_4} \left(\frac{d_1 + d_2 + d_3}{d_4} \right)^{\eta_i} \frac{(\eta_i + 1)^{\eta_i + 1}}{\eta_i^{\eta_i}} \quad (16)$$

In view of (14) and (16), we have

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_1 + d_2 + d_3}{d_4} \right)^{\eta_i} \frac{(\eta_i + 1)^{\eta_i + 1}}{\eta_i^{\eta_i}} \leq 1 \quad (17)$$

which is contrary to the assumption (i), hence every solution of (1) oscillates.

Case 2 $k_0 > 0$, $l_0 = 0$: By the monotonicity of $\omega(x, y)$ and Lemma 2.1, we obtain

$$\begin{aligned}
& \frac{d_2\omega(x+a, y)}{d_4\omega(x, y)} - 1 \\
& \leq \frac{d_1\omega(x+a, y+b) + d_2\omega(x+a, y) + d_3\omega(x, y+b)}{d_4\omega(x, y)} - 1 \\
& \leq -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \frac{\omega(x - k_i a, y - \sigma_i)}{\omega(x, y)} \\
& = -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \frac{\omega(x, y - \sigma_i)}{\omega(x, y)} \prod_{j=1}^{k_i} \frac{\omega(x - ja, y - \sigma_i)}{\omega(x - ja + a, y - \sigma_i)} \\
& < -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \prod_{j=1}^{k_i} \frac{\omega(x - ja, y - \sigma_i)}{\omega(x - ja + a, y - \sigma_i)}
\end{aligned} \tag{18}$$

Let

$$S(x, y) = \frac{\omega(x, y)}{\omega(x+a, y)}, \tag{19}$$

It is easy to see that $S(x, y)$ is bounded, in particular $S(x, y) > 1$, thus

$$\frac{d_2}{d_4 S(x, y)} - 1 < -\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \prod_{j=1}^{k_i} S(x - ja, y - \sigma_i) \tag{20}$$

Denote

$$\xi = \lim_{x, y \rightarrow \infty} \inf S(x, y), \quad \text{then } \xi \in [1, +\infty). \tag{21}$$

Now from (20) and (21), it follows that

$$\frac{d_2}{d_4 \xi} \leq 1 - \frac{1}{d_4} \sum_{i=1}^n q_i \xi^{k_i}, \tag{22}$$

Obviously

$$\xi > \frac{d_2}{d_4}, \quad \frac{\sum_{i=1}^n q_i \xi^{k_i+1}}{d_4 \xi - d_2} \leq 1 \tag{23}$$

Let

$$f(\xi) = \frac{\xi^{k_i+1}}{d_4 \xi - d_2}, \tag{24}$$

Similar to the case 1, we have

$$\min\{f(\xi)\} = \frac{1}{d_4} \left(\frac{d_2}{d_4}\right)^{k_i} \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}} \tag{25}$$

Substituting (25) into (23), it follows that

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_2}{d_4}\right)^{k_i} \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}} \leq 1 \tag{26}$$

which is contrary to the assumption (ii) of theorem 2.2, hence every solution of equation (1) oscillates.

Case 3 $k_0 = 0, l_0 > 0$, The proof is similar to that of $k_0 > 0, l_0 = 0$, and thus, is omitted.

Case 4 $k_0 = 0, l_0 = 0$, Similarly, we have

$$\begin{aligned} 0 &< \frac{d_1\omega(x+a, y+b) + d_2\omega(x+a, y) + d_3\omega(x, y+b)}{d_4\omega(x, y)} \\ &\leq 1 - \frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \frac{\omega(x - \tau_i, y - \sigma_i)}{\omega(x, y)}, \end{aligned} \quad (27)$$

Due to

$$\frac{\omega(x - \tau_i, y - \sigma_i)}{\omega(x, y)} > 1, \quad (28)$$

we have

$$\frac{1}{d_4} \sum_{i=1}^n Q_i(x, y) \leq 1. \quad (29)$$

Which implies that

$$\frac{1}{d_4} \sum_{i=1}^n q_i \leq 1.$$

This contradicts the assumption (iv), so every solution of the equation (1) oscillates. The proof is complete.

3 Example

To illustrate the applications of Theorem 2.1, we consider the following two examples

Example 1 Consider the partial difference equation

$$\begin{aligned} A(x+2\pi, y+2\pi) + A(x+2\pi, y) + A(x, y+2\pi) - A(x, y) \\ + p(x, y)A(x-\pi, y-3\pi) = 0, \end{aligned} \quad (30)$$

Where $p(x, y) = \frac{7}{2} + \sin x + \sin y$. From the equation (30), we have

$$\begin{aligned} u = 1, k = 0, l = 1, d_1 = d_2 = d_3 = d_4 = 1 \\ q = \min_{\substack{x \leq z \leq x+2\pi \\ y \leq w \leq y+2\pi}} p(z, w) = \frac{3}{2}. \end{aligned} \quad (31)$$

Then

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_3}{d_4}\right)^{l_i} \frac{(l_i+1)^{l_i+1}}{l_i^{l_i}} = \frac{1}{1} \cdot \frac{3}{2} \cdot 1^1 \cdot \frac{(1+1)^{1+1}}{1^1} = 6 > 1, \quad (32)$$

According to the assumption (i) of Theorem 2.1, every solution of (30) oscillates.

If we apply the Cor2.2 of [9], first we must calculate the parameter e^{λ^*} which should satisfy

$$d_4(1 - e^{\lambda^*}) = \sum_{i=1}^n q_i e^{-(k_i+l_i)\lambda^*} \left(\frac{d_2}{d_4}\right)^{k_i} \left(\frac{d_3}{d_4}\right)^{l_i}. \quad (33)$$

From (31), we have

$$e^{\lambda^*} = 1 - \frac{3}{2}e^{\lambda^*} \quad (34)$$

but from [9] we find that in order to get (34), the following condition is necessary

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_2}{d_4}\right)^{k_i} \left(\frac{d_3}{d_4}\right)^{l_i} < 1 \quad (35)$$

Substituting (31) into the left side of (35), we have

$$\frac{1}{1} \cdot \frac{3}{2} \cdot \left(\frac{1}{1}\right)^0 \cdot \left(\frac{1}{1}\right)^1 = \frac{3}{2} > 1$$

Obviously the condition (35) does not hold, So we can not use cor2.2 of [9] to discuss the oscillation of equation (30). Hence the theorem 2.1 provides a complement for that of paper[9].

Example 2 Consider the partial difference equation:

$$\begin{aligned} &A(x + 2\pi, y + 2\pi) + A(x + 2\pi, y) + A(x, y + 2\pi) - A(x, y) \\ &+ p_1(x, y)A(x - \pi, y - 3\pi) + p_2(x, y)A(x - 3\pi, y - 5\pi) = 0, \end{aligned} \quad (36)$$

Where $p_1(x, y) = \frac{11}{5} + \sin x + \sin y$, $p_2(x, y) = \frac{5}{2} + \sin x - \sin y$. It is easy to see:

$$\begin{aligned} n = 2, \quad k_1 = 0, l_1 = 1, \quad k_2 = 1, l_2 = 1, \quad d_1 = d_2 = d_3 = d_4 = 1, \\ q_1 = \min_{\substack{x \leq z \leq x + 2\pi \\ y \leq w \leq y + 2\pi}} p_1(z, w) = \frac{1}{5} \\ q_2 = \min_{\substack{x \leq z \leq x + 2\pi \\ y \leq w \leq y + 2\pi}} p_2(z, w) = \frac{1}{2} \end{aligned} \quad (37)$$

By calculating, the equation (36) satisfies the assumption (iii) of theorem 2.1:

$$\frac{1}{d_4} \sum_{i=1}^n q_i \left(\frac{d_3}{d_4}\right)^{l_i} \frac{(l_i + 1)^{l_i+1}}{l_i^{l_i}} = \frac{1}{1} \cdot \frac{1}{5} \cdot \left(\frac{1}{1}\right)^1 \cdot \frac{(1+1)^{1+1}}{1^1} + \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{(2+1)^{2+1}}{2^2} = \frac{77}{40} > 1 \quad (38)$$

Thus the solution of (36) oscillates.

Now we use the oscillation criteria of paper [7] to discuss the oscillation of (36), From [7], we know that if

$$\frac{(1 - e^{\lambda^*})e^{\lambda^*}(\frac{d_1 d_4}{d_2 d_3}e^{\lambda^*} + 2)}{1 - (1 - e^{\lambda^*})e^{\lambda^*}(\frac{d_1 d_4}{d_2 d_3}e^{\lambda^*} + 2)} + (1 - e^{\lambda^*}) > 1. \quad (39)$$

Where e^{λ^*} satisfies (33), then every solution of (1) oscillates.

But in view of (33), we have

$$1 - e^{\lambda^*} = \frac{1}{5} \cdot e^{-\lambda^*} + \frac{1}{2} \cdot e^{-3\lambda^*},$$

Then

$$10e^{4\lambda^*} - 10e^{3\lambda^*} + 2e^{2\lambda^*} + 5 = 0. \quad (40)$$

It is a higher-order equation about e^{λ^*} , So it is difficult to use condition (39) to discuss the oscillation of (36). Obviously theorem 2.1 of this paper is easier to be verified than that of paper[7].

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