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Estimating the Time-to-Failure Distribution of a Linear Degradation Model Using a Bayesian Approach

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Abstract

In this paper, we consider two types of failure data namely grouped and non-grouped, which can be motivated from a linear mixed degradation model. We propose a Bayesian approach to estimate the parameters of the time-to-failure distribution and its percentiles. A simulation study conducted to study the performance of the proposed method showed that in terms of the mean squares error and the length of the bootstrap confidence intervals, the behavior of the proposed method is satisfactory. Also, it showed that the Bayesian approach with non-grouped data is better than the Bayesian approach with a grouped data. Application to a real data set is given.

Keywords: Reliability; Bayesian; Degradation; Grouped data; Non-grouped data; Bootstrap Confidence Interval

1 Introduction

In measuring the reliability of a product, the observed quantity of interest is usually the time-to-failure. Recently, with today's high technology, some life tests result in

no or very few failures by the end of the test, thus it is difficult to assess the reliability by the traditional reliability analysis.
In the literature, Lu and Meeker (1993) used a nonlinear mixed- effects model with random effect parameters that follow a multivariate normal distribution. They used the two-stage method to estimate the model parameters that leads to estimate the time–to-failure distribution. Wu and Shao (1999) established the asymptotic properties of the ordinary and weighted least squares estimators under the nonlinear mixed-effect degradation model. Alodat and Al-Haj Ebrahem (2007) used the ranked set sampling to estimate the parameters of the time-to-failure distribution of a linear degradation model. Al-Haj Ebrahem (2007) estimated the variance components of accelerated degradation models.

Etro and Giorgio (2002) presented a generalized practical Bayesian estimator of the parameters of the Weibull distribution and the inverse power law model under accelerated test. Gebraeel et al. (2005) used a Bayesian method to update the random parameters of an exponential degradation model. Al-Hussaini and Abdel-Hamid (2004) used a Bayesian approach to estimate the parameters of the time-to-failure distribution which is considered as a mixture of two Weibull components. Robinson and Crowder (2000) described a Bayesian approach to estimate and predict the time-to-failure distribution. For more details see Al-Haj Ebrahem and Higgins (2005), Meeker and Escobar (1998), Bagdonavicius and Nikulin (2000), Wang and Daescu (2002) and Hamada (2005).

From the above literature, it is known that some times the time-to-failure of a unit can't be observed exactly, but it can be observed up to intervals. Many studies of degradation models interpolate these time-to-failure data. In this study, we will analyze the linear degradation model without interpolating the time-to-failure data, but we will treat it as a grouped data.

This paper is organized as follows. In Section 2, we present the time-to-failure distribution which can be motivated from a linear mixed degradation model. In Section 3, we propose a Bayesian approach to estimate the parameters of the time-to-failure distribution and its percentiles using non grouped data, while in Section 4 we estimate the parameters of the time-to-failure distribution and its percentiles using grouped data. Simulation study and results are presented in Section 5. An application to real data set is presented in Section 6. Conclusions are discussed in Section 7.

2. Time-to-Failure Distribution and its Percentiles

Consider the following linear mixed degradation model,

$$y_{ij} = \phi + \theta t_j + \varepsilon_{ij}, \tag{1}$$

where y_{ij} is the degradation for unit i at time t_j, ϕ is a fixed effect parameter, θ is a random effect parameter which is assumed to have an exponential distribution with mean μ and ε_{ij}'s are the random error terms which are assumed to be independent and identically distributed with zero mean and constant variance σ_ε^2, i =1, 2,…, n, j = 1, 2,…, m, where n is the number of tested units, m is the number of observations measured for each unit. We assume ε_{ij}'s and θ are independent.

In degradation analysis, we say that the failure occurs at time T when the degradation of a unit reaches a critical degradation level D.

To obtain the time-to-failure distribution of T, let $D = \phi + \theta T$, so under the assumption that θ has an exponential distribution with mean μ, the cumulative distribution of T, $F_T(x)$ is given by,

$$F_T(x) = \exp\left(\frac{\phi - D}{\mu x}\right), \qquad \mu > 0,\ \phi < D\ ,\ x > 0.$$

Hence, the probability density function of T is given by,

$$f_T(x) = \frac{D - \phi}{\mu x^2}\exp\left(\frac{\phi - D}{\mu x}\right), \quad \mu > 0, \phi < D\ ,\ x > 0 \tag{2}$$

Thus, the $100p^{\text{th}}$ percentile t_p of the time-to-failure distribution is given by,

$$t_p = \frac{D - \phi}{-\mu \ln p}.$$

3. A Bayesian Approach with Non Grouped Data

Consider the model defined in equation (1). We assume that ϕ has a uniform prior distribution $\pi_1(\phi)$ and μ has an inverse gamma prior distribution $\pi_2(\mu)$,

where,

$$\pi_1(\phi) = \begin{cases} \dfrac{1}{D} & 0 < \phi < D, \\ 0 & o.w. \end{cases}$$

and

$$\pi_2(\mu) = \begin{cases} \dfrac{\exp\left(\dfrac{-1}{\beta\mu}\right)}{\Gamma(\alpha)\beta^{\alpha}\mu^{\alpha+1}} & \mu > 0, \\ 0 & o.w. \end{cases}$$

We choose the uniform prior distribution $\pi_1(\phi)$, since our model assumes that the failure occurs when the degradation of a unit reaches a critical degradation level *D* and this is valid when the value of the intercept ϕ is less than *D*, also from probability density function of *T* given in equation (2) we should have $0 < \phi < D$. We choose an inverse gamma prior distribution $\pi_2(\mu)$, since the slope of a linear degradation model will be positive and the inverse gamma distribution is natural conjugate prior for μ, which also quite flexible.

Let $x_1, x_2, \ldots, x_n$ be an observed random sample of failure times from the probability density function given in equation (2). The likelihood function of $\underline{x} = (x_1, x_2, \ldots, x_n)$ is given by,

$$L(\underline{x}, \phi, \mu) = \left(\frac{D-\phi}{\mu}\right)^n \prod_{i=1}^{n} \frac{1}{x_i^2} \exp\left(\frac{\phi - D}{\mu} \sum_{i=1}^{n} \frac{1}{x_i}\right).$$

The posterior density of ϕ and μ given $\underline{x}$ is given by,

$$\pi(\phi, \mu \mid \underline{x}) \propto L(\underline{x}, \phi, \mu)\pi_1(\phi)\pi_2(\mu)$$
$$= c\, \frac{(D-\phi)^n}{\mu^{n+\alpha+1}} \exp\left(\frac{1}{\mu}\left((\phi - D)\sum_{i=1}^{n} \frac{1}{x_i} - \frac{1}{\beta}\right)\right), \qquad 0 < \phi < D, \quad \mu > 0.$$

The derivation of *c* is given in Appendix A.

Under the squared error loss function, the Bayesian estimators for ϕ and μ are, respectively, $\hat{\phi}_B$ and $\hat{\mu}_B$,

where,

$$\hat{\phi}_B = \int_0^D \phi\, p(\phi \mid \underline{x})\, d\phi \quad , \qquad \textbf{(3)}$$

$$\hat{\mu}_B = \int_0^\infty \mu \, p(\mu \mid \underline{x}) d\mu \quad , \qquad \textbf{(4)}$$

$p(\phi \mid \underline{x})$ and $p(\mu \mid \underline{x})$ are the marginal posterior density of ϕ and μ respectively.

The derivation of $\hat{\phi}_B$ and $\hat{\mu}_B$ are given in Appendix A.

After computing $\hat{\phi}_B$ and $\hat{\mu}_B$, we can estimate the $100p^{\text{th}}$ percentile t_p of the time-to-failure distribution by,

$$\hat{t}_{pB} = \frac{D - \hat{\phi}_B}{-\hat{\mu}_B \ln p}. \qquad \textbf{(5)}$$

4. A Bayesian Approach with Grouped Data

The time interval $[0, \infty)$ will be partitioned into $(k+1)$ subintervals, $I_1, I_2, \ldots, I_{k+1}$, where,

$I_j = [\,(j-1)\,\delta\,, j\,\delta\,)$, $j = 1, 2, .., k$, $I_{k+1} = [\,k\,\delta\,, \infty\,)$ and δ is the length of the first k subintervals. Since our model assumes the linear degradation path, then each unit will fail in one and only one of these subintervals.

Let z_j be the number of time-to-failures occur within the subinterval I_j, $j = 1, 2, \ldots, k+1$. The probability that a random unit fails in the subinterval I_j is given by,

$$p_j = F_T(j\delta) - F_T((j-1)\delta) = \exp\left(\frac{\phi - D}{j\delta\mu}\right) - \exp\left(\frac{\phi - D}{(j-1)\,\delta\mu}\right), \; j = 2,3,\ldots,k,$$

$$p_1 = F_T(\delta) = \exp\left(\frac{\phi - D}{\delta\mu}\right)$$

and

$$p_{k+1} = 1 - F_T(k\delta) = 1 - \exp\left(\frac{\phi - D}{k\delta\mu}\right) \qquad \textbf{(6)}$$

Thus, z = ($z_1, z_2, .., z_{k+1}$) is distributed as Multinomial (n; $p_1, p_2, \ldots, p_{k+1}$), with $\sum_{j=1}^{k+1} z_j = n$, the number of sampled units.

The likelihood function of $\underline{z} = (z_1, z_2, \ldots, z_{k+1})$ is given by,

$$L(\underline{z},\phi,\mu)=\frac{n!}{z_1!z_2!...z_{k+1}!}\prod_{j=1}^{k+1}p_j^{z_j}$$

Let $A=\dfrac{n!}{z_1!\,z_2!\,...z_{k+1}!}$,

Thus, the likelihood function of $\underline{z}$ is given by,

$$L(\underline{z},\phi,\mu)=A\left[\exp\left(\frac{\phi-D}{\delta\mu}\right)\right]^{z_1}\left[\prod_{j=2}^{k}\left\{\exp\left(\frac{\phi-D}{j\delta\mu}\right)-\exp\left(\frac{\phi-D}{(j-1)\delta\mu}\right)\right\}^{z_j}\right]\left[1-\exp\left(\frac{\phi-D}{k\delta\mu}\right)\right]^{z_{k+1}},$$

$$=A\exp\left(\frac{\phi-D}{\delta\mu}\sum_{j=1}^{k}\frac{z_j}{j}\right)\left[\prod_{j=2}^{k}\left\{1-\exp\left(\frac{\phi-D}{(j-1)j\delta\mu}\right)\right\}^{z_j}\right]\left[1-\exp\left(\frac{\phi-D}{k\delta\mu}\right)\right]^{z_{k+1}}.$$

In this section, we will propose a Bayesian approach to estimate the model parameters ϕ and μ using a grouped data. Assume ϕ has a uniform prior distribution $\pi_1(\phi)$ on (0, D) and μ has an inverse gamma prior distribution $\pi_2(\mu)$ with shape parameter α and scale parameter β, the same priors given in Section 3. The posterior density function of ϕ and μ given $\underline{z}$ is given by,

$$\pi(\phi,\mu|\underline{z})\propto L(\underline{z},\phi,\mu)\pi_1(\phi)\pi_2(\mu)$$

$$=\frac{c^*}{\mu^{\alpha+1}}\exp\left(\frac{-1}{\beta\mu}\right)\left[\exp\left(\frac{\phi-D}{\delta\mu}\sum_{j=1}^{k}\frac{z_j}{j}\right)\right]\times$$
$$\left[\sum_{i_2=0}^{z_2}\cdots\sum_{i_{k+1}=0}^{z_{k+1}}\left\{\left[\prod_{l=2}^{k+1}\binom{z_l}{i_l}\right]\left[(-1)^{\Sigma_{l=2}^{k+1}i_l}\right]\left[\exp\left(\frac{\phi-D}{\delta\mu}\left(\sum_{l=2}^{k}\frac{i_l}{(l-1)l}+\frac{i_{k+1}}{k}\right)\right)\right]\right\}\right].$$

The derivation of c^* is given in Appendix B.

Under the squared error loss function the Bayesian estimators for ϕ and μ are $\hat{\phi}_G$ and $\hat{\mu}_G$, respectively,

where,

$$\hat{\phi}_G = \int_0^D \phi \, p(\phi \mid \underline{z}) \, d\phi \qquad \textbf{(7)}$$

and $$\hat{\mu}_G = \int_0^\infty \mu \, p(\mu \mid \underline{z}) \, d\mu \qquad \textbf{(8)}$$

$p(\phi \mid \underline{z})$ and $p(\mu \mid \underline{z})$ are the marginal posterior density of ϕ and μ respectively.

The derivation of $\hat{\phi}_G$ and $\hat{\mu}_G$ are given in Appendix B.

After computing $\hat{\phi}_G$ and $\hat{\mu}_G$, we can estimate the $100p^{\text{th}}$ percentile t_p of the time-to-failure distribution by,

$$\hat{t}_{pG} = \frac{D - \hat{\phi}_G}{-\hat{\mu}_G \ln p} \qquad \textbf{(9)}$$

5. Simulation Study and Results

To study the performance of the estimators $\hat{\phi}_B$, $\hat{\mu}_B$, $\hat{t}_{pB}$, $\hat{\phi}_G$, $\hat{\mu}_G$ and $\hat{t}_{pG}$, we conducted a simulation study. The mean bias (MB) and the mean squares error (MSE) of the estimators for p = {0.05, 0.1, 0.15} with number of iterations l = 200000 are computed. Simulation indices are: ϕ = 1.5, μ = 3, k = 5, 7, 9, δ = 0.75, 1, 1.25, 2 and D = 5.

To construct a 95% Bootstrap confidence interval for the time-to-failure percentiles using a non grouped data, we used the following steps:

1. Generate a random sample of sample size n = 20, 40 or 60 from the probability density function given in equation (2).
2. Select a sample of size n with replacement from the sample generated in step 1.
3. Compute $\hat{t}_{pB}$ as given in equation (5) for p = 0.05, 0.1, 0.15.
4. Repeat steps 2 and 3 for B times where B = 20000 and let $\hat{t}_{pB}(1)$, $\hat{t}_{pB}(2)$, …, $\hat{t}_{pB}(B)$ be the ordered value of $\hat{t}_{pB}$'s.
5. The 95% Bootstrap confidence interval is (L_B , U_B), where
   $L_B = \hat{t}_{pB}(0.025B)$ and $U_B = \hat{t}_{pB}(0.975B)$.

Similar steps will be used to construct a 95% bootstrap C.I for the time-to-failure percentiles using a grouped data. Simulation results are presented in Tables 1- 4.

Table 1: MB and MSE of $\hat{\phi}_B$, $\hat{\mu}_B$, $\hat{t}_{pB}$ Using Non Grouped Data

Simulation Setting	Parameter	MB	MSE
n = 10, α =5, β =1/12	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.38401 -0.306698 0.010169 0.013231 0.016059	0.292706 0.214334 0.011178 0.018921 0.027874
n = 15, α =5, β =1/12	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.365319 -0.306195 0.009345 0.012159 0.014757	0.244291 0.187717 0.008493 0.014375 0.021176
n = 20, α =5, β =1/12	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.357398 -0.303319 0.007932 0.010319 0.012525	0.217566 0.168951 0.006766 0.011453 0.016871

Table 2: A 95% Bootstrap C.I Using Non Grouped Data

n	t_p	A 95% Bootstrap C.I	Length of C.I
20	$t_{0.05}$ = 0.5007 $t_{0.1}$ = 0.6514 $t_{0.15}$ = 0.7907	(0.2910, 0.6452) (0.3787, 0.8394) (0.4596, 1.0188)	0.3542 0.4608 0.5592
40	$t_{0.05}$ = 0.5007 $t_{0.1}$ = 0.6514 $t_{0.15}$ = 0.7907	(0.3115, 0.6215) (0.4052, 0.8085) (0.4918, 0.9813)	0.3099 0.4033 0.4895
60	$t_{0.05}$ = 0.5007 $t_{0.1}$ = 0.6514 $t_{0.15}$ = 0.7907	(0.4127, 0.6661) (0.5370, 0.8666) (0.6517, 1.0518)	0.2534 0.3296 0.4001

Table 3: MB and MSE of $\hat{\phi}_G$, $\hat{\mu}_G$, $\hat{t}_{pG}$ Using a Grouped Data

Simulation Setting	Parameter	MB	MSE
$k = 5$, $\delta = 2$, $n = 10$, $\alpha = 5$, $\beta = 1/12$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.567609 -0.210507 -0.019247 -0.025040 -0.030392	0.63447 0.227966 0.016398 0.027756 0.040888
$k = 5$, $\delta = 2$, $n = 15$, $\alpha = 5$, $\beta = 1/12$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.506522 -0.230992 -0.012547 -0.016324 -0.019812	0.495692 0.206522 0.013265 0.022453 0.033076
$k = 5$, $\delta = 2$, $n = 20$, $\alpha = 5$, $\beta = 1/12$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.471655 -0.243477 -0.008609 -0.011201 -0.013595	0.422869 0.174515 0.011421 0.019333 0.028479
$k = 5$, $\delta = 2$, $n = 20$, $\alpha = 5$, $\beta = 1/14$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.192991 -0.086218 -0.000706 -0.000919 -0.001116	0.284928 0.194581 0.007523 0.012735 0.01876
$k = 7$, $\delta = 2$, $n = 20$, $\alpha = 5$, $\beta = 1/14$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.194589 -0.084690 -0.000990 -0.001289 -0.001564	0.285016 0.194205 0.007527 0.012741 0.018769
$k = 9$, $\delta = 2$, $n = 20$, $\alpha = 5$, $\beta = 1/14$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.19065 -0.089349 -0.000030 -0.000039 -0.000048	0.289405 0.198316 0.007649 0.012947 0.019072
$k = 5$, $\delta = 0.75$, $n = 10$, $\alpha = 5$, $\beta = 0.125$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.108634 -0.0846556 0.0228952 0.0297874 0.0361537	0.216754 0.0674372 0.0245107 0.0414888 0.0611184
$k = 5$, $\delta = 1$, $n = 10$, $\alpha = 5$, $\beta = 0.125$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.129602 -0.0773405 0.0177719 0.0231218 0.0280635	0.237091 0.0680464 0.0248097 0.0419948 0.0618639
$k = 5$, $\delta = 1.25$, $n = 10$, $\alpha = 5$, $\beta = 0.125$	ϕ μ $t_{0.05}$ $t_{0.1}$ $t_{0.15}$	0.148398 -0.0721267 0.0139955 0.0182086 0.0221002	0.263004 0.070081 0.0257068 0.0435134 0.064101

Table 4: A 95% Bootstrap C.I Using a Grouped Data

n	True value of t_p	95% Bootstrap C.I	Length of C.I
10	$t_{0.05}$ = 0.500712	(0.14032, 0.75219)	0.61187
	$t_{0.1}$ = 0.651442	(0.18256, 0.97863)	0.79607
	$t_{0.15}$ = 0.790672	(0.22158, 1.18779)	0.96621
15	$t_{0.05}$ = 0.500712	(0.24599, 0.72279)	0.47681
	$t_{0.1}$ = 0.651442	(0.32003, 0.94037)	0.62034
	$t_{0.15}$ = 0.790672	(0.38843, 1.14135)	0.75292
20	$t_{0.05}$ = 0.500712	(0.17905, 0.60894)	0.42989
	$t_{0.1}$ = 0.651442	(0.23294, 0.79224)	0.55930
	$t_{0.15}$ = 0.790672	(0.28273, 0.96157)	0.67884

From the above tables, we can conclude that the MB and MSE of the estimators decreases as n increases. MB and MSE of $\hat{t}_{pB}$ and $\hat{t}_{pG}$ increases as p increases. MB and MSE of $\hat{\phi}_G$, $\hat{\mu}_G$ and $\hat{t}_{pG}$ increases as δ increases and as the number of intervals k increases no obvious trend is observed. The length of the bootstrap confidence interval decreases as n increases and increases as p increases.

By comparing the results obtained using non-grouped and a grouped data in terms of MSE's, we conclude that using non-grouped data, the Bayesian estimators are more efficient than the Bayesian estimators obtained using a grouped data. In general, the confidence intervals obtained using a grouped data are wider than those obtained by a non-grouped data.

6. Real Data Application

In this section, we consider the Laser data from Meeker and Escobar (1998), Table C.17, page 642. The Laser data are analyzed using the Bayesian approach with non-grouped and a grouped data. The MSE's of the estimators are obtained using bootstrap method. A 95% confidence interval of the parameters of the time-to-failure distribution and its percentiles are computed.

6.1 Data Description

Over the life of laser devices, degradation causes a decrease in light output. Figure 1 shows the percent increase in operating current, relative to original operating current, over time for a sample of 15 GaAs lasers tested at 80^{o}C. The measurements are taken at time range from 250 to 4000 hours with step equals to 250. In this analysis, we will scale these times by dividing them by 250, failure is assumed to be occurred at a critical degradation level D = 5.

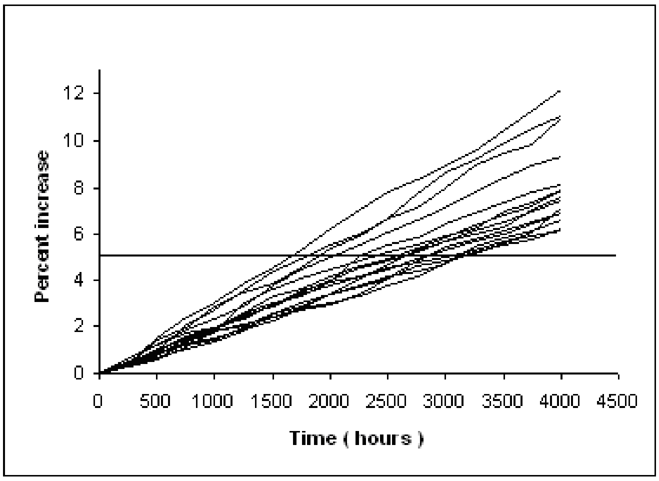

Figure 1: The Percent Increase in Operating Current for GaAs Laser Tested at 80°C

6.2 Data Analysis

Figure 1 shows that the laser data follow a linear degradation path which enables us to use the linear degradation model discussed in Section 2. The estimates of ϕ, μ and t_p, for p = 0.05, 0.1 and 0.15, are obtained using the formulas presented in the previous sections. We assumed $\alpha = 5, \beta = 0.35$, k = 5 and δ =2. The results are presented in Tables (5) and (6), where Table 5 shows the results obtained using non-grouped data and Table 6 shows the results obtained using a grouped data.

Table 5: Results Using Non-Grouped Data

Parameter	Estimate	MSE	95% C.I	Length
ϕ	1.49427	0.00381	(1.37869, 1.61839)	0.2397
μ	0.3912	0.00012	(0.37004, 0.4128)	0.04276
$t_{0.05}$	2.99142	0.01880	(2.73448, 3.26672)	0.53224
$t_{0.1}$	3.89193	0.03182	(3.55764, 4.2501)	0.69246
$t_{0.15}$	4.72374	0.04688	(4.3181, 5.15846)	0.84046

Table 6: Results Using a Grouped Data

Parameter	Estimate	MSE	95% C.I	Length
ϕ	0.83824	0.17460	(0.65837, 1.23227)	0.57389
μ	0.42576	0.00555	(0.35090, 0.47692)	0.12601
$t_{0.05}$	3.26293	0.17149	(2.68429, 4.02931)	1.34503
$t_{0.1}$	4.24517	0.29027	(3.49234, 5.24226)	1.74992
$t_{0.15}$	5.15247	0.42761	(4.23874, 6.36267)	2.12392

From the analysis of Laser degradation data, we see in terms of MSE that the estimates obtained by the Bayesian approach with non-grouped data are more efficient than the estimates obtained by the Bayesian approach with grouped data. Also, we see that the length of the 95% bootstrap confidence interval obtained using the Bayesian approach with non-grouped data is shorter than the length of the 95% bootstrap confidence interval obtained using the Bayesian approach with grouped data.

7. Conclusions

In this section we summarize our finding. In terms of the MSE, the estimators of μ, ϕ and t_p obtained by the Bayesian approach with non-grouped data are more efficient than those obtained by the Bayesian approach with a grouped data. The length of a 95% bootstrap confidence interval of the estimates obtained by the Bayesian approach with non-grouped data is shorter than the length of a 95% bootstrap confidence interval obtained by the Bayesian approach with a grouped data.
The MSE's of the estimators of ϕ, μ and t_p decreases as the sample size increases, MB and MSE of the estimators of t_p increase as p increases and the length of the 95% bootstrap confidence interval decreases as the sample size increases and increases as p increases.

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Appendix A

The derivation of the Bayesian estimators for ϕ and μ using non grouped data,

$$\hat{\phi}_B = \int_0^D \phi\, p(\phi \mid \underline{x}) d\phi$$

$$\hat{\phi}_B = \int_0^D \int_0^\infty \phi\, \pi(\phi, \mu \mid \underline{x}) d\mu\, d\phi,$$

$$= \int_0^D \int_0^\infty c\, \frac{\phi (D-\phi)^n}{\mu^{n+\alpha+1}} \exp\left(\frac{1}{\mu}\left((\phi - D)\sum_{i=1}^n \frac{1}{x_i} - \frac{1}{\beta}\right)\right) d\mu\, d\phi,$$

$$= \frac{c\Gamma(n+\alpha)\, D^{n+2} \beta^{n+\alpha}}{(n+1)(n+2)} \left((n+2)\, H\left(n+1, n+\alpha, n+2, -D\beta \sum_{i=1}^n \frac{1}{x_i}\right) - (n+1)\, H\left(n+2, n+\alpha, n+3, -D\beta \sum_{i=1}^n \frac{1}{x_i}\right)\right)$$

and

$$\hat{\mu}_B = \int_0^\infty \mu\, p(\mu \mid \underline{x}) d\mu$$

$$\hat{\mu}_B = \int_0^D \int_0^\infty \mu\, \pi(\phi, \mu \mid \underline{x})\, d\mu\, d\phi,$$

$$= \int_0^D \int_0^\infty c\, \frac{(D-\phi)^n}{\mu^{n+\alpha}} \exp\left(\frac{1}{\mu}\left((\phi - D)\sum_{i=1}^n \frac{1}{x_i} - \frac{1}{\beta}\right)\right) d\mu\, d\phi,$$

$$= \frac{c\Gamma(n+\alpha-1)\, D^{n+1}\, \beta^{n+\alpha}}{\beta(n+1)(n+2)} \left((n+2)\, H\left(n+1, n+\alpha, n+2, -D\beta \sum_{i=1}^n \frac{1}{x_i}\right) + (n+1)\, D\, \beta \sum_{i=1}^n \frac{1}{x_i} H\left(n+2, n+\alpha, n+3, -D\beta \sum_{i=1}^n \frac{1}{x_i}\right)\right)$$

where,

$$c^{-1} = \int_0^D \int_0^\infty \frac{(D-\phi)^n}{\mu^{n+\alpha+1}} \exp\left(\frac{1}{\mu}\left((\phi - D)\sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\beta}\right)\right) d\mu\, d\phi,$$

$$= \frac{\Gamma(n+\alpha)\ D^{n+1}\ \beta^{n+\alpha}}{n+1} H\left(n+1,\ n+\alpha,\ n+2, -D\beta\sum_{i=1}^{n}\frac{1}{x_i}\right),$$

$$H(a;b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k (b)_k z^k}{(c)_k\ k!}$$

and

$$(a)_k = a(a+1)(a+2)\ldots(a+k\text{-}1)$$

Appendix B

The derivation of the Bayesian estimators for ϕ and μ using a grouped data,

$$\hat{\phi}_G = \int_0^D \phi\, p(\phi \mid \underline{z})\, d\phi$$

$$\hat{\phi}_G = \int_0^D\int_0^\infty \phi\, \pi(\phi, \mu \mid \underline{z})\, d\mu\, d\phi,$$

$$= \sum_{i_2=0}^{z_2} \cdots \sum_{i_{k+1}=0}^{z_{k+1}} \prod_{l=2}^{k+1} \binom{z_l}{i_l} (-1)^{\sum_{l=2}^{k+1} i_l} \int_0^D\int_0^\infty \left[\frac{\phi}{\mu^{\alpha+1}}\ \exp\left(\frac{-1}{\mu}\left(\frac{1}{\beta} + \frac{D-\phi}{\delta}r\right)\right)\right] d\mu\, d\phi,$$

$$= \frac{c^*\delta\, \beta^{\alpha-2}\, \Gamma(\alpha)}{(\alpha-1)\,(\alpha-2)} \times$$

$$\sum_{i_2=0}^{z_2} \cdots \sum_{i_{k+1}=0}^{z_{k+1}} \prod_{l=2}^{k+1} \binom{z_l}{i_l} (-1)^{\sum_{l=2}^{k+1} i_l}\ \frac{1}{r^2}\left[r\alpha\beta D - \delta - 2r\beta D + \delta^{\alpha-1}(r\beta D + \delta)^{2-\alpha}\right]$$

and

$$\hat{\mu}_G = \int_0^\infty \mu\, p(\mu \mid \underline{z})\, d\mu$$

$$\hat{\mu}_G = \int_0^D \int_0^\infty \mu\, \pi(\phi, \mu \mid \underline{z})\, d\mu\, d\phi,$$

$$= \sum_{i_2=0}^{z_2} \cdots \sum_{i_{k+1}=0}^{z_{k+1}} \prod_{l=2}^{k+1} \binom{z_l}{i_l} (-1)^{\sum_{l=2}^{k+1} i_l} \int_0^D \int_0^\infty \left[\frac{\mu}{\mu^{\alpha+1}} \exp\left(\frac{-1}{\mu} \left(\frac{1}{\beta} + \frac{D-\phi}{\delta} r \right) \right) \right] d\mu\, d\phi,$$

$$= \frac{c^* \delta \beta^{\alpha-2} \Gamma(\alpha-1)}{(\alpha-2)} \sum_{i_2=0}^{z_2} \cdots \sum_{i_{k+1}=0}^{z_{k+1}} \prod_{l=2}^{k+1} \binom{z_l}{i_l} (-1)^{\sum_{l=2}^{k+1} i_l} \frac{1}{r} \left[1 - \left(\frac{\delta}{r\beta D + \delta} \right)^{\alpha-2} \right]$$

where,

$$c^{*-1} = \int_0^D \int_0^\infty \sum_{i_2=0}^{z_2} \cdots \sum_{i_{k+1}=0}^{z_{k+1}} \prod_{l=2}^{k+1} \binom{z_l}{i_l} (-1)^{\sum_{l=2}^{k+1} i_l} \frac{1}{\mu^{\alpha+1}} \exp\left(\frac{-1}{\beta\mu} \right) \exp\left(\frac{\phi - D}{\delta\mu} \sum_{j=1}^{k} \frac{z_j}{j} \right) \times$$

$$\exp\left(\frac{\phi - D}{\delta\mu} \left(\sum_{l=2}^{k} \frac{i_l}{(l-1)l} + \frac{i_{k+1}}{k} \right) \right) d\mu\, d\phi,$$

$$= \frac{\Gamma(\alpha)\delta\beta^{\alpha-1}}{(\alpha-1)} \sum_{i_2=0}^{z_2} \cdots \sum_{i_{k+1}=0}^{z_{k+1}} \prod_{l=2}^{k+1} \binom{z_l}{i_l} (-1)^{\sum_{l=2}^{k+1} i_l} \frac{1}{r} \left(1 - \left(\frac{\delta}{rD\beta + \delta} \right)^{\alpha-1} \right)$$

and

$$r = \sum_{j=1}^{k} \frac{z_j}{j} + \sum_{l=2}^{k} \frac{i_l}{(l-1)l} + \frac{i_{k+1}}{k}$$

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