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TL-Moments and L-Moments Estimation for the Generalized Pareto Distribution

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Abstract

In this paper, the trimmed L-moments (TL-moments) and L-moments of the Generalized Pareto distribution (GPD) up to arbitrary order will be derived and used to obtain the first four TL-moments and L-moments. TL-skewness, L-skewness, TL-kurtosis and L-kurtosis are handled for the GPD. Using the first two TL-moments and L-moments, the unknown parameters for the GPD can be estimated. A numerical illustrate for the new results will be given.

Keywords: GPD, TL-moments, L-moments, skewness, kurtosis, Method of TL-moments and L-moments estimation, Beta function, Gamma function, Order statistics

1 Introduction

The method of L-moment estimators have recently appeared. Hosking (1990) gives estimators for log-normal, gamma and generalized extreme value distributions. L-moment estimators for generalized Rayleigh distribution was introduced by Kundu and Raqab (2005). karvanen (2006) applied the method of L-moment estimators to estimate the parameters of polynomial quantile mixture. He introduced the mixture composed of two parametric families, are the normal-polynomial quantile and Cauchy-polynomial quantile. The standard method to compute the L-moment estimators is to equate the sample L-moments with the corresponding population L-moments. A population L-moment L_r is defined to be a certain linear function of the expectations of the order statistics $Y_{1:r}, Y_{2:r}, ..., Y_{r:r}$ in a conceptual random sample of size r from the underlying population. For example, $L_1 = E(Y_{1:1})$, which is the same as the population mean, is defined in terms of a conceptual sample of size $r = 1$, while $L_2 = (1/2)E(Y_{2:2} - Y_{1:2})$, an alternative to the population standard deviation, is defined in terms of a conceptual sample of size $r = 2$. Similarly, the L-moments L_3 and L_4 are alternatives to the un-scaled measures of skewness and kurtosis μ_3 and μ_4 respectively. See Silito (1969). Compared to the conventional moments, L-moments have lower sample variances and are more robust against outliers. Elamir and Seheult (2003) introduced an extension of L-moments called TL-moments. TL-moments are more robust than L-moments and exist even if the distribution does not have a mean, for example the TL-moments are existed for Cauchy distribution. Abdul-Moniem (2007) applied the method of L-moment and TL-moment estimators to estimate the parameters of exponential distribution. The following formula gives the r^{th} TL-moments (see Elamir and Seheult (2003)).

$$L_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E(Y_{r+t-k:r+2t}), \tag{1}$$

where r and t take the values $1, 2, 3, \ldots$. Note that the r^{th} L-moments can be obtained by taking $t = 0$. The GPD is defined by Abd Elfattah et. al (2007). They derived some well know distributions as a special cases from GPD. GPD has the following probability density function form:

$$f(y;\alpha,\theta,\lambda,\delta) = \frac{\delta\alpha}{\theta}\left(\frac{y-\lambda}{\theta}\right)^{\delta-1}\left[1+\left(\frac{y-\lambda}{\theta}\right)^{\delta}\right]^{-(\alpha+1)};$$
$$y \geq \lambda > 0, \alpha, \theta \& \delta > 0 \tag{2}$$

where θ is the scale parameter, λ is the location parameter and (α, δ) are the shape parameters. The corresponding cumulative distribution function is

$$F(y;\alpha,\theta,\lambda,\delta) = 1 - \left[1+\left(\frac{y-\lambda}{\theta}\right)^{\delta}\right]^{-\alpha}. \tag{3}$$

The main aim of this paper is to derive TL-moments and L-moments of the GPD up to arbitrary order and using it to estimate the unknown parameters. This paper is organized as follows: in Section 2, we introduced population TL-moments and TL-moment estimators for the GPD. The population L-moments and L-moment estimators for the GPD was presented in Section 3. In Section 4, A numerical illustrate for the new results will be given.

2 TL-moments for the GPD

In this section, the population TL-moment of order r for the GPD will be obtained. The sample TL-moments and the TL-moments estimators also discussed.

2.1 Population TL-moments

Using formula (1) and two functions (2) and (3), the TL-moment of order r for the GPD taking the following form

$$L_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{(r+2t)!}{(r+t-k-1)!(t+k)!}(I),$$

where

$$I = \int_\lambda^\infty \frac{y[1-[1+(\frac{y-\lambda}{\theta})^\delta]^{-\alpha}]^{r+t-k-1}\frac{\delta\alpha}{\theta}(\frac{y-\lambda}{\theta})^{\delta-1}}{[1+\frac{y-\lambda}{\theta})^\delta]^{\alpha(t+k+1)+1}}dy$$

By expanding $[1-[1+(\frac{y-\lambda}{\theta})^\delta]^{-\alpha}]^{r+t-k-1}$ binomially, we get

$$I = \sum_{j=0}^{r+t-k-1}\binom{r+t-k-1}{j}(-1)^j\int_\lambda^\infty \frac{y\frac{\delta\alpha}{\theta}(\frac{y-\lambda}{\theta})^{\delta-1}}{[1+(\frac{y-\lambda}{\theta})^\delta]^{\alpha(t+k+j+1)+1}}dy$$

let $z=(\frac{y-\lambda}{\theta})^\delta$, this led to $y=\theta z^{\frac{1}{\delta}}+\lambda$ and $|J|=\frac{\theta}{\delta(\frac{y-\lambda}{\theta})^{\delta-1}}$, then

$$I = \sum_{j=0}^{r+t-k-1}\binom{r+t-k-1}{j}(-1)^j\alpha\int_0^\infty \frac{\theta z^{\frac{1}{\delta}}+\lambda}{[1+z]^{\alpha(t+k+j+1)+1}}dy$$

$$= \sum_{j=0}^{r+t-k-1}\binom{r+t-k-1}{j}(-1)^j\alpha[\theta\beta(1+\frac{1}{\delta},\alpha(t+k+j+1)-\frac{1}{\delta})$$

$$+\frac{\lambda}{\alpha(t+k+j+1)}]$$

The $L_r^{(t)}$ becomes

$$L_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{(r+2t)!}{(r+t-k-1)!(t+k)!}\sum_{j=0}^{r+t-k-1}\binom{r+t-k-1}{j}$$

$$(-1)^j\alpha[\theta\beta(1+\frac{1}{\delta},\alpha(t+k+j+1)-\frac{1}{\delta})+\frac{\lambda}{\alpha(t+k+j+1)}] \qquad (4)$$

where $r,t=1,2,3,\ldots$. Here, we take $t=1$ (see Elamir and Seheult (2003)) then equation (4) becomes

$$L_r^{(1)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\left(\begin{array}{c} r-1 \\ k \end{array}\right)\frac{(r+2)!}{(r-k)!(1+k)!}\sum_{j=0}^{r-k}\left(\begin{array}{c} r-k \\ j \end{array}\right)$$

$$(-1)^j\alpha[\theta\beta(1+\frac{1}{\delta},\alpha(k+j+2)-\frac{1}{\delta})+\frac{\lambda}{\alpha(k+j+2)}] \tag{5}$$

where $r = 1, 2, 3, \ldots; \alpha, \lambda, \delta$ and $\theta > 0$. The first four TL-moments can be obtained by taking $r = 1, 2, 3$ and 4 in (5) as follows

$$L_1^{(1)} = \frac{\theta\Gamma(1+\frac{1}{\delta})[3\Gamma(3\alpha)\Gamma(2\alpha-\frac{1}{\delta})-2\Gamma(2\alpha)\Gamma(3\alpha-\frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(3\alpha)}+\lambda \tag{6}$$

$$L_2^{(1)} = \frac{3\theta\Gamma(1+\frac{1}{\delta})[\Gamma(4\alpha)\Gamma(2\alpha-\frac{1}{\delta})+\Gamma(2\alpha)\Gamma(4\alpha-\frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(4\alpha)}$$

$$-\frac{6\theta\Gamma(1+\frac{1}{\delta})\Gamma(3\alpha-\frac{1}{\delta})}{\Gamma(3\alpha)} \tag{7}$$

$$L_3^{(1)} = \frac{10\theta\Gamma(1+\frac{1}{\delta})[\Gamma(3\alpha)\Gamma(2\alpha-\frac{1}{\delta})-4\Gamma(2\alpha)\Gamma(3\alpha-\frac{1}{\delta})]}{3\Gamma(2\alpha)\Gamma(3\alpha)}$$

$$+\frac{10\theta\Gamma(1+\frac{1}{\delta})[5\Gamma(5\alpha)\Gamma(4\alpha-\frac{1}{\delta})-2\Gamma(4\alpha)\Gamma(5\alpha-\frac{1}{\delta})]}{3\Gamma(4\alpha)\Gamma(5\alpha)} \tag{8}$$

and

$$L_4^{(1)} = \frac{15\theta\Gamma(1+\frac{1}{\delta})[\Gamma(4\alpha)\Gamma(2\alpha-\frac{1}{\delta})+15\Gamma(2\alpha)\Gamma(4\alpha-\frac{1}{\delta})]}{4\Gamma(2\alpha)\Gamma(4\alpha)}$$

$$+\frac{35\theta\Gamma(1+\frac{1}{\delta})[\Gamma(5\alpha)\Gamma(6\alpha-\frac{1}{\delta})-3\Gamma(6\alpha)\Gamma(5\alpha-\frac{1}{\delta})]}{2\Gamma(5\alpha)\Gamma(6\alpha)}$$

$$-\frac{25\theta\Gamma(1+\frac{1}{\delta})\Gamma(3\alpha-\frac{1}{\delta})}{\Gamma(3\alpha)} \tag{9}$$

The TL-skewness (∇_3) and TL-kurtosis (∇_4) will be

$$\nabla_3 = \frac{L_3^{(1)}}{L_2^{(1)}} = \frac{10\Gamma(4\alpha)[\Gamma(3\alpha)\Gamma(2\alpha-\frac{1}{\delta})-4\Gamma(2\alpha)\Gamma(3\alpha-\frac{1}{\delta})]}{9\Psi(\alpha,\delta)}$$

$$+\frac{10\Gamma(2\alpha)\Gamma(3\alpha)[5\Gamma(5\alpha)\Gamma(4\alpha-\frac{1}{\delta})-2\Gamma(4\alpha)\Gamma(5\alpha-\frac{1}{\delta})]}{9\Gamma(5\alpha)\Psi(\alpha,\delta)} \tag{10}$$

and

$$\nabla_4 = \frac{L_4^{(1)}}{L_2^{(1)}} = \frac{5\Gamma(4\alpha)[3\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta}) - 20\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})]}{12\Psi(\alpha, \delta)}$$
$$+ \frac{15\Gamma(2\alpha)\Gamma(3\alpha)[15\Gamma(5\alpha)\Gamma(4\alpha - \frac{1}{\delta}) - 14\Gamma(4\alpha)\Gamma(5\alpha - \frac{1}{\delta})]}{12\Gamma(5\alpha)\Psi(\alpha, \delta)}$$
$$+ \frac{70\Gamma(2\alpha)\Gamma(3\alpha)\Gamma(4\alpha)\Gamma(6\alpha - \frac{1}{\delta})}{12\Gamma(6\alpha)\Psi(\alpha, \delta)} \qquad (11)$$

where

$$\Psi(\alpha, \delta) = \Gamma(4\alpha)[\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta}) - 2\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})]$$
$$+\Gamma(2\alpha)\Gamma(3\alpha)\Gamma(4\alpha - \frac{1}{\delta})$$

3 Sample TL-moments and TL-moment estimators

TL-moments can be estimated from a sample as linear combination of order statistics. Elamir and Seheult (2003) present the following estimator for sample TL-moments:

$$l_r^{(t)} = \frac{1}{r} \sum_{i=t+1}^{n-t} \frac{\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t-k-1} \binom{n-i}{t+k}}{\binom{n}{r+2t}} x_{i:n} \qquad (12)$$

where $a \geq b$ for all $\binom{a}{b}$ and $x_{i:n}$ denotes the i^{th} order statistic in a sample of size n. From (6), (7) and (12) with α and δ are known and $t = 1$, we can get the TL-moment estimator for $\theta(\hat{\theta}_{TL})$ and $\lambda(\hat{\lambda}_{TL})$ as follows

$$l_1^{(1)} = \frac{6}{n(n-1)(n-2)} \sum_{i=2}^{n-1} (i-1)(n-i) x_{i:n}$$
$$= \frac{\hat{\theta}_{TL}\Gamma(1 + \frac{1}{\delta})[3\Gamma(3\alpha)\Gamma(2\alpha - \frac{1}{\delta}) - 2\Gamma(2\alpha)\Gamma(3\alpha - \frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(3\alpha)} + \hat{\lambda}_{TL}, \qquad (13)$$

$$l_2^{(1)} = \frac{12}{n(n-1)(n-2)(n-3)}\{\sum_{i=3}^{n-1}\binom{i-1}{2}\binom{n-i}{1}x_{i:n}$$

$$-\sum_{i=2}^{n-2}\binom{i-1}{1}\binom{n-i}{2}x_{i:n}\}$$

$$= \frac{3\hat{\theta}_{TL}\Gamma(1+\frac{1}{\delta})[\Gamma(4\alpha)\Gamma(2\alpha-\frac{1}{\delta})+\Gamma(2\alpha)\Gamma(4\alpha-\frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(4\alpha)}$$

$$-\frac{6\hat{\theta}_{TL}\Gamma(1+\frac{1}{\delta})\Gamma(3\alpha-\frac{1}{\delta})}{\Gamma(3\alpha)} \quad (14)$$

By solving equations (13) and (14), we get

$$\hat{\theta}_{TL} = l_2^{(1)} \div \{\frac{3\Gamma(1+\frac{1}{\delta})[\Gamma(4\alpha)\Gamma(2\alpha-\frac{1}{\delta})+\Gamma(2\alpha)\Gamma(4\alpha-\frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(4\alpha)}$$

$$-\frac{6\Gamma(1+\frac{1}{\delta})\Gamma(3\alpha-\frac{1}{\delta})}{\Gamma(3\alpha)}\}, \quad (15)$$

and

$$\hat{\lambda}_{TL} = l_1^{(1)} - \frac{\hat{\theta}_{TL}\Gamma(1+\frac{1}{\delta})[3\Gamma(3\alpha)\Gamma(2\alpha-\frac{1}{\delta})-2\Gamma(2\alpha)\Gamma(3\alpha-\frac{1}{\delta})]}{\Gamma(2\alpha)\Gamma(3\alpha)} \quad (16)$$

4 L-moments for the GPD

In this section, the population L-moment of order r for the GPD as a special case from formula (4) will be introduced. Sample L-moments and L-moments estimators also studied.

4.1 Population L-moments

Here, the population L-moment of order r for the GPD as a special case form (4) by taking $t = 0$ will be

$$L_r = \sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}^2\sum_{j=0}^{r-k-1}\binom{r-k-1}{j}(-1)^j$$

$$\alpha[\theta\beta(1+\frac{1}{\delta},\alpha(k+j+1)-\frac{1}{\delta})+\frac{\lambda}{\alpha(k+j+1)}] \quad (17)$$

The first four L-moments can be obtained by taking $r = 1, 2, 3$ and 4 in (17) as follows

$$L_1 = \frac{\theta\Gamma(1+\frac{1}{\delta})\Gamma(\alpha-\frac{1}{\delta})}{\Gamma(\alpha)} + \lambda, \quad (18)$$

$$L_2 = \frac{\theta\Gamma(1+\frac{1}{\delta})[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]}{\Gamma(\alpha)\Gamma(2\alpha)} \tag{19}$$

$$L_3 = \frac{\theta\Gamma(1+\frac{1}{\delta})[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - 3\Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]}{\Gamma(\alpha)\Gamma(2\alpha)} + \frac{2\theta\Gamma(1+\frac{1}{\delta})\Gamma(3\alpha-\frac{1}{\delta})}{\Gamma(3\alpha)} \tag{20}$$

and

$$L_4 = \frac{\theta\Gamma(1+\frac{1}{\delta})[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - 6\Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]}{\Gamma(\alpha)\Gamma(2\alpha)} + \frac{5\theta\Gamma(1+\frac{1}{\delta})[2\Gamma(4\alpha)\Gamma(3\alpha-\frac{1}{\delta}) - \Gamma(3\alpha)\Gamma(4\alpha-\frac{1}{\delta})]}{\Gamma(3\alpha)\Gamma(4\alpha)} \tag{21}$$

The L-skewness (τ_3) and L-kurtosis (τ_4) will be

$$\tau_3 = \frac{L_3}{L_2} = \frac{\Gamma(2\alpha)\Gamma(3\alpha)\Gamma(\alpha-\frac{1}{\delta}) - 3\Gamma(\alpha)\Gamma(3\alpha)\Gamma(2\alpha-\frac{1}{\delta})}{\Gamma(3\alpha)[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]} + \frac{2\Gamma(\alpha)\Gamma(2\alpha)\Gamma(3\alpha-\frac{1}{\delta})}{\Gamma(3\alpha)[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]} \tag{22}$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - 6\Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})}{\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})} + \frac{5\Gamma(\alpha)\Gamma(2\alpha)[2\Gamma(4\alpha)\Gamma(3\alpha-\frac{1}{\delta}) - \Gamma(3\alpha)\Gamma(4\alpha-\frac{1}{\delta})]}{\Gamma(3\alpha)\Gamma(4\alpha)[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]} \tag{23}$$

4.2 Sample L-moments and L-moment estimators

Sample L-moments can be estimated from (12) by taking $t = 0$ as follows

$$l_r = \frac{1}{r}\sum_{i=1}^{n} \frac{\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}}{\binom{n}{r}} x_{i:n} \tag{24}$$

where $x_{i:n}$ as above. From (18), (19) and (24) with α and δ are known, the L-moment estimator for $\theta(\hat{\theta}_L)$ and $\lambda(\hat{\lambda}_L)$ will be

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} x_{i:n} = \bar{x} = \frac{\hat{\theta}_L\Gamma(1+\frac{1}{\delta})\Gamma(\alpha-\frac{1}{\delta})}{\Gamma(\alpha)} + \hat{\lambda}_L, \tag{25}$$

and

$$l_2 = \frac{2}{n(n-1)} \sum_{i=1}^{n} (i-1) x_{i:n} - \bar{x}$$

$$= \frac{\hat{\theta}_L \Gamma(1+\frac{1}{\delta})[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]}{\Gamma(\alpha)\Gamma(2\alpha)} \quad (26)$$

By solving equations (25) and (26), we get

$$\hat{\theta}_L = \frac{l_2 \Gamma(\alpha)\Gamma(2\alpha)}{\Gamma(1+\frac{1}{\delta})[\Gamma(2\alpha)\Gamma(\alpha-\frac{1}{\delta}) - \Gamma(\alpha)\Gamma(2\alpha-\frac{1}{\delta})]} \quad (27)$$

and

$$\hat{\lambda}_L = l_1 - \frac{\hat{\theta}_L \Gamma(1+\frac{1}{\delta})\Gamma(\alpha-\frac{1}{\delta})}{\Gamma(\alpha)} \quad (28)$$

5 A numerical illustration

By generating samples of size 10(10)40 with 10000 replications. Applying the program of Mathcad (2001), the estimates and their mean square error (MSE) of the unknown parameters θ and λ using equations (15), (16), (27) and (28) are computed. Table (1) presents the estimates of θ and λ and its MSEs using the exact value of $\lambda = 2$ with different values of $\theta = 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4$ and 1.6.

Table(1) Estimates and MSEs of θ and λ

θ		n=10	n=20	n=30	n=40
0.2	$\hat{\lambda}_L$	2.0003(0.0023)	2.0004(0.0011)	1.9998(0.0008)	2.0003(0.0006)
	$\hat{\lambda}_{TL}$	2.0002(0.0017)	2.0003(0.0007)	1.9999(0.0005)	2.0002(0.0003)
	$\hat{\theta}_L$	0.1988(0.0139)	0.1989(0.0068)	0.2003(0.0051)	0.1991(0.0035)
	$\hat{\theta}_{TL}$	0.1990(0.0082)	0.1990(0.0037)	0.2000(0.0025)	0.2000(0.0017)
0.4	$\hat{\lambda}_L$	1.9992(0.0088)	1.9998(0.0041)	1.9998(0.0032)	2.0007(0.0022)
	$\hat{\lambda}_{TL}$	1.9989(0.0072)	1.9993(0.0028)	1.9998(0.0018)	2.0004(0.0013)
	$\hat{\theta}_L$	0.4044(0.0563)	0.4001(0.0261)	0.4003(0.0202)	0.3982(0.0141)
	$\hat{\theta}_{TL}$	0.4050(0.0354)	0.4010(0.0148)	0.4000(0.0098)	0.3990(0.0069)
0.6	$\hat{\lambda}_L$	1.9989(0.0199)	1.9995(0.0101)	1.9997(0.0065)	2.0005(0.0049)
	$\hat{\lambda}_{TL}$	1.9984(0.0162)	1.9993(0.0065)	1.9999(0.0041)	2.0007(0.0029)
	$\hat{\theta}_L$	0.6066(0.1266)	0.6001(0.0637)	0.6003(0.0418)	0.5992(0.0317)
	$\hat{\theta}_{TL}$	0.6070(0.0796)	0.6000(0.0342)	0.6000(0.0221)	0.5990(0.0159)
0.8	$\hat{\lambda}_L$	2.0005(0.0354)	2.0001(0.0189)	2.0005(0.0115)	2.0006(0.0087)
	$\hat{\lambda}_{TL}$	1.9988(0.0284)	2.0011(0.0114)	1.9996(0.0072)	2.0009(0.0052)
	$\hat{\theta}_L$	0.7964(0.2187)	0.7998(0.1212)	0.7984(0.0744)	0.7989(0.0563)
	$\hat{\theta}_{TL}$	0.8010(0.1400)	0.7980(0.0609)	0.8000(0.0392)	0.7990(0.0282)

Table(1) Continued

θ		n=10	n=20	n=30	n=40
1.0	$\hat{\lambda}_L$	2.0006(0.0553)	2.0012(0.0256)	1.9992(0.0183)	1.9979(0.0152)
	$\hat{\lambda}_{TL}$	1.9984(0.0444)	1.9992(0.0181)	1.9992(0.0112)	1.9993(0.0083)
	$\hat{\theta}_L$	0.9955(0.3418)	0.9968(0.1647)	1.0038(0.1215)	1.0058(0.0973)
	$\hat{\theta}_{TL}$	1.0010(0.2188)	1.0010(0.0952)	1.0040(0.0631)	1.0020(0.0463)
1.2	$\hat{\lambda}_L$	1.9930(0.1185)	1.9999(0.0392)	1.9995(0.0288)	1.9975(0.0218)
	$\hat{\lambda}_{TL}$	2.0022(0.0633)	2.0011(0.0251)	1.9999(0.0161)	1.9991(0.0119)
	$\hat{\theta}_L$	1.2193(0.7124)	1.2012(0.2482)	1.2024(0.1846)	1.2070(0.1402)
	$\hat{\theta}_{TL}$	1.1980(0.3094)	1.1990(0.1325)	1.2020(0.0886)	1.2030(0.0667)
1.4	$\hat{\lambda}_L$	1.9919(0.1612)	1.9992(0.0568)	1.9989(0.0346)	1.9982(0.0280)
	$\hat{\lambda}_{TL}$	2.0025(0.0862)	1.9974(0.0355)	1.9981(0.0222)	1.9978(0.0161)
	$\hat{\theta}_L$	1.4225(0.9697)	1.3996(0.3612)	1.4018(0.2251)	1.4027(0.1783)
	$\hat{\theta}_{TL}$	1.3970(0.4212)	1.4020(0.1871)	1.4030(0.1215)	1.4030(0.0882)
1.6	$\hat{\lambda}_L$	2.0022(0.1476)	2.0002(0.0683)	2.0003(0.0482)	2.0026(0.0388)
	$\hat{\lambda}_{TL}$	1.9969(0.1123)	1.9980(0.0444)	2.0011(0.0282)	2.0013(0.0208)
	$\hat{\theta}_L$	1.5986(0.9117)	1.5991(0.4397)	1.5952(0.3081)	1.5912(0.2451)
	$\hat{\theta}_{TL}$	1.6100(0.5492)	1.6040(0.2408)	1.5940(0.1523)	1.5960(0.1126)

The MSEs are reported within brackets against each estimates.

From Table (1), one can show that

- The values of MSEs decrease as n increases.
- The values of MSEs for $\hat{\lambda}_{TL}$ and $\hat{\theta}_{TL}$ are smaller than the corresponding values for $\hat{\lambda}_L$ and $\hat{\theta}_L$.
- The values of MSEs for $\hat{\lambda}_{TL}, \hat{\theta}_{TL}, \hat{\lambda}_L$ and $\hat{\theta}_L$ increase as the exact value of θ increases.

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