Applied Mathematical Sciences, Vol. 3, 2009, no. 2, 53 - 59

Note on the Optimality of Some Two-Way Elimination of Heterogeneity Designs

Henryk Brzeskwiniewicz and Jolanta Krzyszkowska [1]

Department of Mathematical and Statistical Methods
Poznań University of Life Sciences
Wojska Polskiego 28, 60-637 Poznań, Poland

Abstract

Simple lower bounds for A-, D-, E- and L-efficiency of some two-way elimination of heterogeneity designs are derived. The bounds are obtained on the basis of the eigenvalues of information matrix for treatment effects.

Mathematics Subject Classification: 62K10, 62K15

Keywords: A-efficiency, D-efficiency; E-efficiency, L-efficiency, Eigenvalues, Information matrix, Lower bound, Two-way elimination of heterogeneity design

1 Introduction and notation

Any arrangement of the v treatments in the b_1 rows and b_2 columns is called a two-way elimination of heterogeneity design. Let $\mathbf{r} = (r_1, ..., r_v)'$, $\mathbf{k}_1 = (k_{1_1}, ..., k_{1_{b_1}})'$ and $\mathbf{k}_2 = (k_{2_1}, ..., k_{2_{b_2}})'$ denote a vector of treatment replications, a vector of row sizes and a vector of column sizes, respectively. Let $\mathbf{R}$ and $\mathbf{K}_1$ and $\mathbf{K}_2$ be the diagonal matrices with the successive elements of $\mathbf{r}$, $\mathbf{k}_1$ and $\mathbf{k}_2$ on their diagonals. Moreover, let $\mathbf{N}_1$ be the $v \times b_1$ treatment-row incidence matrix, let $\mathbf{N}_2$ be the $v \times b_2$ treatment-column incidence matrix. The $\mathbf{C}$-matrices of the two related subdesigns are

$$\mathbf{C}_h = \mathbf{R} - \mathbf{N}_h \mathbf{K}_h^{-1} \mathbf{N}_h' \quad (1)$$

with $h = 1$ for treatment-row subdesigns and $h = 2$ for treatment-column subdesigns.

[1] krzyszko@up.poznan.pl

In this paper we consider designs with information matrix for the treatment effects defined by [3]:

$$\mathbf{C} = \xi_1 \mathbf{C}_1 + \xi_2 \mathbf{C}_2 - \xi_0 \mathbf{C}_0, \tag{2}$$

where $\xi_1 > 0$, $\xi_2 > 0$, and $\xi_0 > 0$, $\mathbf{C}_0 = \mathbf{R} - \mathbf{r}\mathbf{r}'/n$ and n is the number of experimental units. Let $D\left(n, v, b_1, b_2, r_{min}, r_{max}, k_{1_{max}}, k_{2_{max}}, h\right)$ denotes the collection of two-way elimination of heterogeneity designs whose $\mathbf{C}$-matrix admit a representation in the form (2), where $r_{min} = \min\limits_{1 \le i \le v} r_i$, $r_{max} = \max\limits_{1 \le i \le v} r_i$, $k_{1_{max}} = \max\limits_{1 \le j \le b_1} k_{1_j}$, $k_{2_{max}} = \max\limits_{1 \le j \le b_2} k_{2_j}$, and h is the rank of $\mathbf{C}$ ($h \le v-1$, if $h = v-1$ then a design is said to be connected).

It should be noted that in the theory of experimental designs, A-, D- and E-optimality is often considered. For example, [8] and [11] considered A-, D- and E-optimality for designs for quadratic and cubic growth curve models and for designs for polynomial growth models with auto-correlated errors, respectively. A-optimal chemical balance weighing designs and A-optimal designs under a quadratic growth curve model in the transformed time interval are presented respectively by [6] and [9]. On the E-optimality of nested row-column designs, of designs in irregular BIB settings, of designs with three treatments and of designs under an interference model are considered by [2], [12], [13] and [7], respectively. Note that A-, D-, E- and L-efficiency for block designs are presented by [5].

2 Results

For a design $d \in D\left(n, v, b_1, b_2, r_{min}, r_{max}, k_{1_{max}}, k_{2_{max}}, h\right)$ let $0 = \mu_{d_0} \le \mu_{d_1} \le \ldots \le \mu_{d_v - 1}$ denote eigenvalues of its $\mathbf{C}$-matrix. Define

$$\phi_A(d) = \sum_{i=v-h}^{v-1} \mu_{d_i}^{-1},\ \phi_D(d) = \prod_{i=v-h}^{v-1} \mu_{d_i}^{-1},\ \phi_E(d) = \mu_{d_{v-h}},\ \phi_L(d) = \sum_{i=v-h}^{v-1} \mu_{d_i}. \tag{3}$$

A design d is A- or D-optimal if it minimises the $\phi_A(d)$ or $\phi_D(d)$ values among all the possible from some class of designs. A design d is E- or L-optimal if it maximises the $\phi_E(d)$ or $\phi_L(d)$ values among all the possible from some class of designs. The A-, D-, E- and L-efficiency of a design d is defined to be

$$\begin{aligned} e_A(d) &= \frac{\phi_A\left(d_A^*\right)}{\phi_A(d)}, & e_D(d) &= \frac{\phi_D\left(d_D^*\right)}{\phi_D(d)}, \\ e_E(d) &= \frac{\phi_E\left(d\right)}{\phi_{E/R}(d_E^*)}, & e_L(d) &= \frac{\phi_L\left(d\right)}{\phi_L(d_L^*)}, \end{aligned} \tag{4}$$

where d_A^*, d_D^*, d_E^* and d_L^* are A-, D-, E- and L-optimal designs, respectively.

One problem with these definitions is that optimal designs are known only for some special cases. Therefore, in the next section simple lower bounds of (4) will be given as some conservative measures of the efficiencies of design d.

2.1 Lower bounds of e_A and e_D

Note that for $d \in D\left(n, v, b_1, b_2, r_{min}, r_{max}, k_{1_{max}}, k_{2_{max}}, h\right)$ from (1) and (2) we have
$\mu_{d_{v-1}} = \mathbf{p}'\mathbf{C}\mathbf{p} = \xi_1\mathbf{p}'\mathbf{R}\mathbf{p} + \xi_2\mathbf{p}'\mathbf{R}\mathbf{p} - \xi_1\mathbf{p}'\mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1'\mathbf{p} - \xi_2\mathbf{p}'\mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2'\mathbf{p} - \xi_0\mathbf{p}'\mathbf{R}\mathbf{p} + \xi_0\mathbf{p}'\frac{\mathbf{r}'\mathbf{r}}{n}\mathbf{p} \leq \left(\xi_1 + \xi_2\right) r_{\max} + \xi_0 r_{\max}^2 \mathbf{p}'\mathbf{1}\mathbf{1}'\mathbf{p} = \left(\xi_1 + \xi_2\right) r_{\max}$.
From above and (3) we have

$$\phi_A\left(d_A^*\right) \geq \frac{h}{\left(\xi_1 + \xi_2\right) r_{\max}} \quad \text{and} \quad \phi_D\left(d_D^*\right) \geq \frac{1}{\left(\left(\xi_1 + \xi_2\right) r_{\max}\right)^h}. \tag{5}$$

2.2 Another lower bounds of e_A and e_D

From (1) and (2) we have

$$\begin{aligned} &tr\left(\mathbf{C}\right) = \\ &\xi_1 \sum_{i=1}^{v}\left(r_i - \sum_{j=1}^{b_1}\frac{n_{1_{ij}}^2}{k_{1_j}}\right) + \xi_2 \sum_{i=1}^{v}\left(r_i - \sum_{j=1}^{b_2}\frac{n_{2_{ij}}^2}{k_{2_j}}\right) - \xi_0 \sum_{i=1}^{v}\left(r_i - \frac{r_i^2}{n}\right) \leq \\ &\xi_1 n\left(1 - \frac{1}{k_{1_{\max}}}\right) + \xi_2 n\left(1 - \frac{1}{k_{2_{\max}}}\right) - \xi_0 n + \xi_0\frac{v r_{\max}^2}{n} = \\ &n\left(\xi_1\left(1 - \frac{1}{k_{1_{\max}}}\right) + \xi_2\left(1 - \frac{1}{k_{2_{\max}}}\right) - \xi_0\left(1 - \frac{v r_{\max}^2}{n}\right)\right) = t \end{aligned} \tag{6}$$

and

$$\bar{\mu}_d = \frac{\sum_{i=v-h}^{v-1}\mu_{d_i}}{h} \leq \frac{t}{h} \tag{7}$$

Observe that

$$\sum_{i=v-h}^{v-1}\mu_{d_i}^{-1} \geq \frac{h}{\bar{\mu}_d} \quad \text{and} \quad \prod_{i=v-h}^{v-1}\mu_{d_i}^{-1} \geq \frac{1}{\bar{\mu}_d^h} \tag{8}$$

From (7) and (8) we have, in particular,

$$\phi_A\left(d_A^*\right) \geq \frac{h^2}{t} \quad \text{and} \quad \phi_D\left(d_D^*\right) \geq \left(\frac{h}{t}\right)^h \tag{9}$$

From (5) and (9) follows that

$$e_A\left(d_A^*\right) \geq \max\left\{\frac{h}{(\xi_1 + \xi_2) r_{max}}, \frac{h^2}{t}\right\}, \; e_D\left(d_D^*\right) \geq \max\left\{\frac{1}{\left((\xi_1 + \xi_2) r_{max}\right)^h}, \left(\frac{h}{t}\right)^h\right\}$$

which leads (see (4)) to

$$e_A(d) \geq \frac{\max\left\{\frac{h}{((\xi_1+\xi_2)r_{max}}, \ \frac{h^2}{t}\right\}}{\phi_A(d)}, \quad e_D(d) \geq \frac{\max\left\{\frac{h}{(\xi_1+\xi_2)r_{max}}, \ \left(\frac{h}{t}\right)^h\right\}}{\phi_D(d)} \tag{10}$$

and therefore two efficiency lower bounds of e_A and e_D are defined as

$$e'_A(d) = \frac{\max\left\{\frac{h}{(\xi_1+\xi_2)r_{max}}, \ \frac{h^2}{t}\right\}}{\phi_A(d)}, \quad e'_D(d) = \frac{\max\left\{\frac{1}{((\xi_1+\xi_2)r_{max})^h}, \ \left(\frac{h}{t}\right)^h\right\}}{\phi_D(d)}. \tag{11}$$

2.3 Lower bounds of e_E

Let block designs d_h, $h = 1{,}2$ with information matrix $\mathbf{C}_h$ (see 1) contain a block which consists of m common distinct treatments and $2 \leq m \leq v-1$. We assume, by eventually relabelling the treatments and reshuffling the blocks, that the first block consists of m distinct treatments with numbers $1, ..., m$. Then

$$\mu_{d_1} \leq \frac{v}{m(v-m)} \left(\xi_1 P_{d_1}(m) + \xi_2 P_{d_2}(m) - \xi_0 P_{d_0}(m)\right) = P_d(m), \tag{12}$$

where $P_{d_h}(m) = \sum_{i=1}^m r_i \left(1 - \frac{1}{k_{h_{\max}}}\right) - (k_{h_1} - 1)$ [4] and principal minor of $\mathbf{C}_0$ is at least from $P_{d_0}(m) = \sum_{i=1}^m r_i - \frac{(mr_{max})^2}{n}$, because $\sum_{i=1}^m r_i - \frac{1}{n}\sum_{i,j=1}^m r_i r_j \leq \sum_{i=1}^m r_i - \frac{(mr_{max})^2}{n}$. Note that in the paper of [4] we have weak inequalities $P_{d_0}(m) = \sum_{i=1}^m r_i - \frac{\left(\sum_{i=1}^m r_i\right)^2}{n}$. On the other hand

$$\mu_{d_1} \leq \frac{v}{v-1} \left(\xi_1 T_{d_1} + \xi_2 T_{d_2} - \xi_0 T_{d_0}\right) = T_d, \tag{13}$$

where $T_{d_h} = r_{min}\left(1 - \frac{1}{k_{2_{max}}}\right)$ [4] and $T_{d_0} = r_{max}\left(1 - \frac{r_{min}}{n}\right)$ because i-th diagonal element of $\mathbf{C}_0$ is equal to $r_i - \frac{r_i^2}{n}$, and $r_i - \frac{r_i^2}{n} = r_i\left(1 - \frac{r_i}{n}\right) \leq r_{max}\left(1 - \frac{r_{min}}{n}\right)$. Note that in the paper of [4] we have weak inequalities $T_{d_0} = r_{min}\left(1 - \frac{r_{min}}{n}\right)$. From (12) and (13) we have

$$\phi_E\left(d_E^*\right) \leq \min\{P_d(m), \ T_d\}. \tag{14}$$

From (14) and (4) it follows that

$$e_E\left(d\right) \geq \frac{\phi_E(d)}{\min\{P_d(m), \ T_d\}} \tag{15}$$

and therefore the lower bound of e_E is defined as

$$e'_E\left(d\right) = \frac{\phi_E(d)}{\min\{P_d(m), \ T_d\}}. \tag{16}$$

2.4 Lower bounds of e_L

From (3) and (6) we have

$$\phi_L\left(d_L^*\right) \leq t, \tag{17}$$

where t is defined by (6). Formulae (17) and (4) imply that

$$e_L(d) \geq \frac{\phi_L(d)}{t} \tag{18}$$

and therefore the lower bound of e_L is defined as

$$e_L'(d) = \frac{\phi_L(d)}{t}. \tag{19}$$

3 Examples

We consider A-, D-, E-, L-efficiency of the designs shown in Tables 1 and 2.

Table 1

Rows	Columns 1	2	3	4
1	1	2	4	3
2	7	8	5	6
3	5	6	1	2
4	3	4	8	7

Table 2

Rows	Columns 1	2	3	4	5	6	7
1		3	5		2		
2			4	6		3	
3				5	7		4
4	5				6	1	
5		6				7	2
6	3		7				1
7	2	4		1			

In the case of Table 1, $d \in D(16, 8, 4, 4, 2, 2, 4, 4, 7)$ with $\xi_1 = \xi_2 = \xi_0 = 1$ and $\mu_{d_1} = \mu_{d_2} = \mu_{d_3} = \mu_{d_4} = 1$, $\mu_{d_5} = \mu_{d_6} = \mu_{d_7} = 2$, [10]. We calculate $\phi_.(d)$ occurring in (3) as: $\phi_A(d) = 4 \cdot 1 + 3 \cdot \frac{1}{2} = \frac{11}{2}$, $\phi_D(d) = 1^4 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, $\phi_E(d) = 1$ and $\phi_L(d) = 4 \cdot 1 + 3 \cdot 2 = 10$. But d_1 and d_2 have no block with m distinct treatments, then we calculate only T_d occurring in (13) as $T_d = \frac{8}{7} \cdot \left(2\left(1 - \frac{1}{4}\right) + 2\left(1 - \frac{1}{4}\right) - 2\left(1 - \frac{2}{16}\right)\right) = \frac{10}{7}$. Hence according to formulae (11), (16) and (17) we obtain:
$e_A'(d) = \frac{\max\left\{\frac{7}{4}, \frac{49}{40}\right\}}{\frac{11}{2}} = \frac{7}{22} \approx 0.32$ $e_D'(d) = \frac{\max\left\{\left(\frac{1}{4}\right)^7, \left(\frac{7}{40}\right)^7\right\}}{\frac{11}{2}} = \frac{1}{2^{11}} \approx 0.00049$ $e_E'(d) = \frac{1}{\frac{10}{7}} = 0.7$ and $e_L'(d) = \frac{10}{16 \cdot \frac{5}{2}} = 0.25$. We have obtained a high $e_E'(d)$ value, therefore we consider that the discussed design is close to the E-optimal

design, but this design is far from the A-, D- and L-optimal design.

In Table 2, $d \in D(21,7,7,7,3,3,3,3,6)$ with $\xi_1 + \xi_2 = 1$, $\xi_0 = \frac{4}{9}$ and $\mu_{d_1} = \mu_{d_2} = \mu_{d_3} = \mu_{d_4} = \mu_{d_5} = \mu_{d_6} = 1$ [1]. From (3) we have: $\phi_A(d) = \phi_L(d) = 6$ and $\phi_D(d) = \phi_E(d) = 1$. But d_1 and d_2 have block with $m = 3$ distinct treatments, then we calculate $P_d(3)$ and T_d occurring in (12) and (13), respectively; $P_d(3) = \frac{7}{3\cdot 4}\left(4 - \frac{16}{7}\right) = 1$ and $T_d = \frac{7}{6}\left(2 - \frac{8}{7}\right) = 1$. From (11), (16) and (17) we obtain:
$e'_A(d) = \frac{\max\left\{2, \frac{9}{14}\right\}}{6} = \frac{1}{3} \approx 0.33$, $e'_D(d) = \frac{\max\left\{\left(\frac{1}{2}\right)^6, \left(\frac{9}{14}\right)^6\right\}}{1} = \frac{1}{2^6} \approx 0.0156$, $e'_E(d) = \frac{1}{\min\{1,1\}} = 1$ and $e'_L(d) = \frac{6}{21\cdot\left(\frac{2}{3}+\frac{8}{9}\right)} = \frac{18}{119} \approx 0.15$. The discussed design is far from the A-, D- and L-optimal design and it is E-optimal design ($e'_E(d) = 1$).

References

[1] H.L. Agrawal, Some systematic methods of constructions of design for two-way elimination of heterogeneity, *Calcutte Statist. Assoc. Bull.,* 15 (1966), 93–108.

[2] S. Bagchi, Two infinite series of E-optimal nested row-column designs, *J. Statist. Plann. Inference,* 52 (1996), 353–357.

[3] J. Berube and G.P.H. Styan, Decomposable three-way layouts, *J. Statist. Plann. Inference,* 36(2-3) (1993), 311–322.

[4] H. Brzeskwiniewicz, On the E-optimality of some two-way elimination of heterogeneity designs, *Statist. Probab. Lett.,* 25 (1995), 9–13.

[5] H. Brzeskwiniewicz, On the A-, D-, E- and L-efficiency of block designs, *Statist. Probab. Lett.,*30 (1996), 205–209.

[6] B. Ceranka, M. Graczyk and K. Katulska, On certain A-optimal chemical balance weighing designs, *Comput. Statist. Data Anal.,* 51 (2007), 5821–5827.

[7] K. Filipiak and R. Różański, E-optimal designs under an interference model, *Biom. Lett.,* 42(2) (2005), 133–142.

[8] K. Filipiak and A. Szczepańska, A-, D- and E-optimal designs for quadratic and cubic growth curve models with correlated errors,*Biom. Lett.,* 42(1) (2005), 43–56.

[9] K. Filipiak and A. Szczepańska, A-optimal designs under a quadratic growth curve model in the transformed time interval, *Biom. Lett.,* 44(2) (2007), 333–340.

[10] M. Kozłowska and R. Walkowiak, Properties and constructions of row-and-column designs with adjusted orthogonality, *Coll. Biom.,* 21 (1990), 60–69 (in Polish).

[11] M. Moerbeek, Robustness properties of A-, D- and E-optimal designs for polynomial growth models with autocorrelated errors, *Comput. Statist. Data Anal.,* 48 (2005), 765–778.

[12] J.P. Morgan and B.H. Reck, E-optimal design in irregular BIBD settings, *J. Statist. Plann. Inference,* 137 (2007), 1658–1668.

[13] V. Parvu and J.P. Morgan, E-optimal designs for three treatments, *To appear in J. Statist. Plann. Inference,* (2007), http: designtheory.org/library/preprints/E3multiway.pdf.

Received: October 1, 2008