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The Negative Neumann Eigenvalues of Second Order Differential Equation with Two Turning Points

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Abstract

In this work, we have studied the solution of the second order differential equation with two turning points case. It is well known that the solutions of the equation are obtained by the asymptotic solution. The aim of this article is to show the higher order distribution negative eigenvalues of Sturm-Liouville problem with Neumann boundary conditions in two turning points case.

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1 Introduction

In the case of ordinary differential equations the book by Wasow [6] discusses several aspects with detail and rigour. One of the basic tools in the sturm-liouville theory is the idea of asymptotic of solution, for example see [1]. The start of the systematic investigations of the asymptotic eigenvalues of sturm-Liouville problems has been made in Haupt [3] and Richardson[4]. In [5] we obtained the positive asymptotic eigenvalues for sturm-liouville problems with boundary condition $w'(a) = w'(b) = 0$ with two turning points. The purpose of this paper is to derive asymptotic eigenvalue for differential equation

$$\frac{d^2w}{d\xi^2} = \{u^2(1-\xi^2) + \varphi(\xi)\}w \tag{1}$$

in which the independent variable ξ range over (a, b), and u is a large parameter and the $\varphi(\xi)$ is continuous function on (a, b). We also saw the asymptotic eigenvalues for equation (1), when the boundary conditions are $w(a) = w(b) = 0$

with one turning point (see [2]). In this paper we study the negative asymptotic eigenvalues with Neumann boundary condition $w'(a) = w'(b) = 0$ with two turning points case.

2 Relevant properties of parabolic functions

In this paper we will consider the equation (1). Any particular integral of the equation whereas the general solution of the corresponding homogeneous equation. This section we study the standard form of approximating differential equation

$$\frac{d^2W}{dx^2} = (l - \frac{1}{4}x^2)W \tag{2}$$

when $l > 0$, the turning points are at $x = \pm 2l^{\frac{1}{2}}$ and for large positive l when $l = \frac{1}{2}\mu^2$ and $x = \mu y\sqrt{2}$, by [1] the solutions of (2) are in following form

$$k^{\frac{-1}{2}}(\frac{1}{2}\mu^2)W(\frac{1}{2}\mu^2, \mu y\sqrt{2}) = 2^{\frac{1}{4}}\pi^{\frac{1}{2}}\mu^{\frac{-1}{6}}(\frac{\eta}{y^2-1})^{\frac{1}{4}}B_i(-\mu^{\frac{4}{3}}\eta)(1 + O(u^{-1})) \tag{3}$$

$$k^{\frac{1}{2}}(\frac{1}{2}\mu^2)W(\frac{1}{2}\mu^2, -\mu y\sqrt{2}) = 2^{\frac{1}{4}}\pi^{\frac{1}{2}}\mu^{\frac{-1}{6}}(\frac{\eta}{y^2-1})^{\frac{1}{4}}A_i(-\mu^{\frac{4}{3}}\eta)(1 + O(u^{-1})), \tag{4}$$

where $\eta = \frac{3}{2}\int_1^y(\tau^2 - 1)d\tau^{\frac{2}{3}}$, $k(\frac{1}{2}\mu^2 = \frac{1}{2}e^{-\frac{1}{2}\pi\mu^2} + O\left(e^{-\frac{1}{2}\pi\mu^2}\right)$ and $W(\frac{1}{2}\mu^2, \mu y\sqrt{2})$, $W(\frac{1}{2}\mu^2, -\mu y\sqrt{2})$ are two independent solutions of (2).

3 Derivative of solutions

For next section, in Neumann boundary condition, we need the derivative of solutions. For this purpose, let $\theta = (\frac{\eta}{\xi^2-1})^{\frac{1}{4}}$ then $\frac{d\theta}{d\xi} = \theta' = \frac{1}{4}(\xi^2 - 1)^{\frac{1}{2}}\eta^{\frac{-5}{4}} - \frac{1}{2}(\xi^2 - 1)^{\frac{-5}{6}}\eta^{\frac{1}{4}}$ so the derivative of solutions (3) and (4) are in form of

$$\frac{\partial W(\frac{1}{2}u, \xi\sqrt{2u})}{\partial\xi} = k^{\frac{1}{2}}(\frac{1}{2}u)u^{\frac{-1}{12}}\sqrt[4]{2\pi^2}(\theta' B_i + \theta B_i'), \tag{5}$$

$$\frac{\partial W(\frac{1}{2}u, -\xi\sqrt{2u})}{\partial\xi} = k^{\frac{-1}{2}}(\frac{1}{2}u)u^{\frac{-1}{12}}\sqrt[4]{2\pi^2}(\theta' A_i + \theta A_i').$$

We consider differential equation (1). By [1] for $\xi > 0$ have two solutions in following form

$$w_1(u, \xi) = k^{\frac{-1}{2}}(\frac{u}{2})W(\frac{1}{2}u, \xi\sqrt{2u}) + \epsilon_1, \qquad w_2(u, \xi) = k^{\frac{1}{2}}(\frac{u}{2})W(\frac{1}{2}u, -\xi\sqrt{2u}) + \epsilon_2, \tag{6}$$

where ϵ_1 and ϵ_2 are error terms. So the derivative of this solution are

$$\frac{\partial w_1}{\partial \xi} = u^{\frac{-1}{12}} \sqrt[4]{2\pi^2}(\theta' B_i + \theta B_i') + \epsilon_1, \qquad \frac{\partial w_2}{\partial \xi} = u^{\frac{-1}{12}} \sqrt[4]{2\pi^2}(\theta' A_i + \theta A_i') + \epsilon_2, \quad (7)$$

Lemma 1 *The asymptotic form of* $A_i(u^{\frac{2}{3}}\eta)$, $A_i'(u^{\frac{2}{3}}\eta)$, $B_i(u^{\frac{2}{3}}\eta)$, $B_i'(u^{\frac{2}{3}}\eta)$ *are given by* $(u \longrightarrow \infty, \xi > 0), \alpha = \frac{2}{3}u\eta_b^{\frac{3}{2}} - \frac{\pi}{4}$

$$A_i(-u^{\frac{2}{3}}\eta_b) \sim \frac{M_1 cos\alpha + M_2 sin\alpha}{2\pi^{\frac{1}{2}} u^{\frac{1}{6}} \eta_b^{\frac{1}{4}}}, \quad A_i'(-u^{\frac{2}{3}}\eta_b) \sim u^{\frac{1}{6}} \eta_b^{\frac{1}{4}} \pi^{\frac{-1}{2}} \left\{ M_3 sin\alpha - M_4 cos\alpha \right\}, \tag{8}$$

$$B_i(-u^{\frac{2}{3}}\eta_b) \sim \frac{M_2 cos\alpha - M_1 sin\alpha}{2\pi^{\frac{1}{2}} u^{\frac{1}{6}} \eta_b^{\frac{1}{4}}}, \quad B_i'(-u^{\frac{2}{3}}\eta_b) \sim u^{\frac{1}{6}} \eta_b^{\frac{1}{4}} \pi^{\frac{-1}{2}} \left\{ M_3 cos\alpha + M_4 sin\alpha \right\}. \tag{9}$$

For $\xi < 0$, $a < -1$, $\beta = \frac{2}{3}u\eta_{-a}^{\frac{3}{2}} - \frac{\pi}{4}$ the Airy function are given by,

$$A_i(-u^{\frac{2}{3}}\eta_{-a}) \sim \frac{M_5 cos\beta + M_6 sin\beta}{2\pi^{\frac{1}{2}} u^{\frac{1}{6}} \eta_{-a}^{\frac{1}{4}}}, \quad A_i'(-u^{\frac{2}{3}}\eta_{-a}) \sim u^{\frac{1}{6}} \eta_{-a}^{\frac{1}{4}} \pi^{\frac{-1}{2}} \left\{ M_7 sin\beta - M_8 cos\beta \right\}, \tag{10}$$

$$B_i(-u^{\frac{2}{3}}\eta_{-a}) \sim \frac{M_6 cos\beta - M_5 sin\beta}{2\pi^{\frac{1}{2}} u^{\frac{1}{6}} \eta_{-a}^{\frac{1}{4}}}, B_i'(-u^{\frac{2}{3}}\eta_{-a}) \sim u^{\frac{1}{6}} \eta_{-a}^{\frac{1}{4}} \pi^{\frac{-1}{2}} \left\{ M_7 cos\beta + M_8 sin\beta \right\}, \tag{11}$$

where the M_i are given by

$$M_1 = \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{(\frac{2}{3}u\eta_b^{\frac{3}{2}})^{2s}}, \qquad M_2 = \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{(\frac{2}{3}u\eta_b^{\frac{3}{2}})^{2s+1}},$$
$$M_3 = \sum_{s=0}^{\infty} (-1)^s \frac{V_{2s}}{(\frac{2}{3}u\eta_b^{\frac{3}{2}})^{2s}}, \qquad M_4 = \sum_{s=0}^{\infty} (-1)^s \frac{V_{2s+1}}{(\frac{2}{3}u\eta_b^{\frac{3}{2}})^{2s+1}},$$
$$M_5 = \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{(\frac{2}{3}u\eta_{-a}^{\frac{3}{2}})^{2s}}, \qquad M_6 = \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{(\frac{2}{3}u\eta_{-a}^{\frac{3}{2}})^{2s+1}},$$
$$M_7 = \sum_{s=0}^{\infty} (-1)^s \frac{V_{2s}}{(\frac{2}{3}u\eta_{-a}^{\frac{3}{2}})^{2s}}, \qquad M_8 = \sum_{s=0}^{\infty} (-1)^s \frac{V_{2s+1}}{(\frac{2}{3}u\eta_{-a}^{\frac{3}{2}})^{2s+1}}.$$

We suppose

$$\psi_1 = -M_1 sin\alpha + M_2 cos\alpha, \qquad \psi_2 = -M_3 cos\alpha + M_4 sin\alpha,$$
$$\psi_1' = \alpha'(b)(-M_1 cos\alpha - M_2 sin\alpha), \qquad \psi_2' = \alpha'(b)(-M_3 sin\alpha + M_4 cos\alpha)$$
$$\psi_3 = -M_6 cos\beta - M_5 sin\beta, \qquad \psi_4 = -M_5 cos\beta + M_6 sin\beta,$$
$$\psi_3' = \beta'(a)(-M_5 cos\beta - M_6 sin\beta), \qquad \psi_4' = \beta'(a)(-M_5 sin\beta + M_6 cos\beta),$$
$$\theta = (b^2-1)^{\frac{-1}{4}}, \qquad \gamma = (a^2-1)^{\frac{-1}{4}}.$$

For $\xi < -1$ the solutions of equation (1) are in form of

$$w_1(u,a) = 2^{\frac{1}{4}} \pi^{\frac{1}{2}} u^{-\frac{1}{12}} (a^2-1)^{-\frac{1}{4}} \eta_{-a}^{\frac{1}{4}} \frac{1}{2\pi^{\frac{1}{2}} u^{\frac{1}{6}} \eta_{-a}^{\frac{1}{4}}} (-M_5 \sin\beta + M_6 \cos\beta),$$

$$w_1(u,a) = 2^{-\frac{3}{4}} u^{-\frac{1}{4}} (a^2-1)^{-\frac{1}{4}} (M_6 \cos\beta - M_5 \sin\beta), \tag{12}$$

similarly

$$w_2(u,a) = 2^{-\frac{3}{4}} u^{-\frac{1}{4}} (a^2-1)^{-\frac{1}{4}} (M_5 \cos\beta + M_6 \sin\beta). \tag{13}$$

The derivative of $w_1(u,a)$ and $w_2(u,a)$ are in form of

$$w_1'(u,a) = 2^{-\frac{3}{4}} u^{-\frac{1}{4}} (\gamma'\psi_3 + \gamma\psi_3') \qquad w_2'(u,a) = 2^{-\frac{3}{4}} u^{-\frac{1}{4}} (\gamma'\psi_4 + \gamma\psi_4'), \tag{14}$$

where

$$\beta' = -\frac{2}{3} u (a^2-1) \eta_{-a}^{\frac{1}{2}} = \beta_1 u, \quad \gamma = (a^2-1)^{-\frac{1}{4}}, \quad \psi_3' = -\beta'_{(a)}\psi_4, \quad \psi_4' = \beta'_{(a)}\psi_3. \tag{15}$$

4 Asymptotic of the eigenvalues

Now we will study distribution of eigenvalue of equation (1) with Neumann condition $w'(a) = w'(b) = 0$. The eigenvalue of equation (1) are the zero of $\Delta(u) = 0$ where $\Delta(u)$ is defined as following form

$$\Delta(u) = \begin{vmatrix} w_1'(u,b) & w_2'(u,b) \\ w_1'(u,a) & w_2'(u,a) \end{vmatrix}, \tag{16}$$

therefore we get

$$w_1'(u,b) w_2'(u,a) - w'(u,a) w_2'(u,b) = 0$$

$$\Delta(u) = [\gamma'\psi_3 - \gamma\beta'(a)\psi_4](\theta'\psi_2 + \theta\psi_2') - (\gamma'\psi_4 + \gamma\beta'(a)\psi_3)(\theta'\psi_1 + \theta\psi_1') = 0 \tag{17}$$

$$\begin{aligned} \Delta(u) \quad &= [\gamma'(M_6 cos\beta - M_5 sin\beta) - \gamma\beta'(M_5 cos\beta + M_6 sin\beta)] \\ &\times \left[\theta'(M_3 cos\alpha + M_4 sin\alpha) + \theta\alpha'_{(b)}(-M_3 sin\alpha + M_4 cos\alpha)\right] \\ &- [\gamma'(M_5 cos\beta + M_6 sin\beta) + \gamma\beta'(-M_5 sin\beta + M_4 cos\beta)] \\ &\times [\theta' - (-M_1 sin\alpha + M_2 cos\alpha) + \theta\alpha'(-M_1 cos\alpha - M_2 sin\alpha)] = 0. \end{aligned}$$

Here we have $M_1 = M_5 = u_0 + O(u^{-1})$ and $M_3 = u_0 + O(u^{-1})$

$$M_6 \cos\beta - M_5 \sin\beta = \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{(\frac{2}{3} u \eta_{-a}^{\frac{3}{2}})^{2s+1}} \cos\beta - \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{(\frac{2}{3} u \eta_{-a}^{\frac{3}{2}})^{s}} \sin\beta$$

$$= \Big(\frac{u_1}{\frac{2}{3} u \eta_{-a}^{\frac{2}{3}}} - \frac{u_3}{(\frac{2}{3} u \eta_{-a}^{\frac{2}{3}})} + \ldots\Big) \cos\beta - \Big(u_0 - \frac{u_2}{\frac{2}{3} u \eta_{-a}^{\frac{2}{3}}} + \ldots\Big) \sin\beta = -(u_0 + O(u^{-1})) \sin\beta. \tag{18}$$

Similarly we will have

$$-M_5\cos\beta - M_6\sin\beta = -(u_0 + O(u^{-1}))\cos\beta$$

$$M_2\cos\alpha + M_4\sin\alpha = O(u^{-1})$$

$$-M_3\sin\alpha + M_4\cos\alpha = -(v_0 + O(u^{-1}))\sin\alpha$$

$$M_5\cos\beta + M_6\sin\beta = (u_0 + O(u^{-1}))\cos\beta$$

$$-M_5\sin\beta + M_6\cos\beta = -(u_0 + O(u^{-1}))\sin\beta$$

$$-M_1\sin\alpha + M_2\cos\alpha = -(u_0 + O(u^{-1}))\sin\alpha$$

$$-M_1\cos\alpha - M_2\sin\alpha = -(u_0 + O(u^{-1}))\cos\alpha. \qquad (19)$$

Therefore by substituting (19) in $\Delta(u)$ then we will have

$$\Delta(u) = \Big[(-\gamma'\sin\beta - \gamma\beta_1 u\cos\beta)\,(u_0 + O(u^{-1}))\Big]\Big[O(u^{-1}) - \theta\alpha_1 u(v_0 + O(u^{-1}))\sin\alpha\Big] -$$

$$\Big[(\gamma'\cos\beta - \gamma\beta_1 u\sin\beta)\,(u_0 + O(u^{-1}))\Big]\Big[(-\theta'\sin\alpha - \theta\alpha_1 u\cos\alpha)\,(u_0 + O(u^{-1}))\Big] = 0 \qquad (20)$$

where $\beta' = \beta_1 u$ and $\alpha' = \alpha_1 u$. By deviation (20) to u then

$$[O(u^{-1}) - \gamma\beta_1(u_0 + O(u^{-1}))\cos\beta][O(u^{-2}) - \theta\alpha_1(v_0 + O(u^{-1}))\sin\alpha] -$$

$$[O(u^{-1}) - \gamma\beta_1(u_0 + O(u^{-1}))\sin\beta][O(u^{-2}) - \theta\alpha_1(u_0 + O(u^{-1}))\cos\alpha] = 0 \quad (21)$$

$$O(u^{-1}) + \gamma\theta\beta_1\alpha_1(u_0 + O(u^{-1}))(v_0 + O(u^{-1}))\sin\alpha\cos\beta -$$

$$(O(u^{-1}) + \gamma\theta\beta_1\alpha_1(u_0 + O(u^{-1}))(u_0 + O(u^{-1}))\sin\beta\cos\alpha) = 0. \qquad (22)$$

By deviation from $\alpha_1\beta_1\gamma\theta$ in (22) then we will have

$$[O(u^{-1}) + (u_0 + O(u^{-1}))(v_0 + O(u^{-1}))\sin\alpha\cos\beta - (u_0 + O(u^{-1}))(u_0 + O(u^{-1}))\sin\beta\cos\alpha = 0$$

For $u \longrightarrow \infty$ we have

$$\tan\alpha.\cot\beta = \frac{u_0 + O(u^{-1})}{v_0 + O(u^{-1})} = \frac{M_5}{M_3},$$

since $\tan\beta = \beta + \frac{\beta^3}{3} + \frac{2\beta^5}{15} + \ldots$ then we see that

$$\tan\alpha = \frac{M_5}{M_3}\left[\tan\beta = \frac{M_5}{M_3}\beta + \frac{\beta^3}{3} + \frac{2\beta^5}{15} + \ldots\right],$$

we suppose $\tan\alpha = x$ and we get arctan of two side, therefore

$$\alpha = n\pi + \arctan x = n\pi + \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots \qquad |x| > 1$$

Hence we get

$$\frac{2}{3}u\eta_b^{\frac{3}{2}} - \frac{\pi}{4} = n\pi + \frac{3\pi}{4} - \frac{1}{x}\left(1 + O(x^{-2})\right),$$

at last

$$u_n = \frac{2n\pi + \frac{3\pi}{2}}{\frac{4}{3}\eta_b^{\frac{3}{2}}} - \frac{3}{2\eta_b^{\frac{3}{2}}x}\left(1 + O(x^{-2})\right).$$

Theorem 2 *Under the conditions of the given Neumann problem, the eigenvalues of Neumann at the origin are*

$$u_n = \frac{2n\pi + \frac{3\pi}{2}}{\frac{4}{3}\eta_b^{\frac{3}{2}}} - \frac{3}{2\eta_b^{\frac{3}{2}}x}\left(1 + O(x^{-2})\right).$$

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