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A New Method for Solving of Rectangular Fuzzy Linear System of Equations Based on Greville's Algorithm

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Abstract

Linear systems have important applications to many branches of science and engineering. In many applications, at least some of the parameters of the system are represented by fuzzy rather than crisp numbers. So it is immensely important to develop numerical procedures that would appropriately treat general fuzzy linear systems and solve them. In this paper, we propose a new method for solving fuzzy linear system based on Greville 's algorithm, where the coefficient matrix of the extended system is the rectangular form. The method in detail is discussed and illustrated by solving some numerical examples.

Keywords: fuzzy linear system of equations, Greville 's method, triangular fuzzy number

1. Introduction

Linear system of equations is important for studying and solving a large proportion of the problems in many topics in applied mathematics and engineering. One of the most applications of fuzzy number arithmetic is treating fuzzy linear systems, several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. Friedman et al. [5] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system. After that, in the literature of fuzzy linear system of equations, various methods is proposed to solve these systems [1, 2, 3, 7, 8, 9]. In this

study, we focus on the rectangular fuzzy linear system of equations (FLSE), and propose a new method for solving these systems based on Greville 's algorithm. The structure of this paper is organized as follows:
In Section 2, we state some basic definitions and notations of fuzzy sets and the linear system of equations (FLSE). In Section 3, we describe Greville process to obtain a new method for solving fuzzy linear systems. In Section 4, we propose Greville 's method for solving FLSE. Numerical examples are given in Section 5 to illustrate our method.

2. Preliminaries

In this section we give some fundamental definition, concepts and results concerning to fuzzy linear system (taken from [8]). We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left semicontinuous non-decreasing function over [0, 1],
2. $\overline{u}(r)$ is a bounded left semicontinuous non-increasing function over [0, 1],
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

In particular, if $\underline{u}$, $\overline{u}$ are linear functions we have an important kind of fuzzy numbers entitled triangular fuzzy number. The set of all triangular fuzzy numbers is denoted by E. For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as:

$$\begin{array}{ll} 1. & x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r)). \\ 2. & kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases} \\ 3. & x - y = (\underline{x}(r) - \overline{y}(r), \overline{x}(r) - \underline{y}(r)). \end{array} \tag{1}$$

Definition 1. Two fuzzy numbers $x(r) = (\underline{x}(r), \overline{x}(r))$, $y(r) = (\underline{y}(r), \overline{y}(r))$ are equal, i.e, $x = y$ if and only if

$$\underline{x}(r) = \underline{y}(r), \overline{x}(r) = \overline{y}(r), 0 \leq r \leq 1. \tag{2}$$

Definition 2. The $m \times n$ linear system

$$\begin{cases} a_{11}(\underline{x}_1(r), \overline{x}_1(r)) + ... + a_{1n}(\underline{x}_n(r), \overline{x}_n(r)) = (\underline{y}_1(r), \overline{y}_1(r)), \\ a_{21}(\underline{x}_1(r), \overline{x}_1(r)) + ... + a_{2n}(\underline{x}_n(r), \overline{x}_n(r)) = (\underline{y}_2(r), \overline{y}_2(r)), \\ \quad \vdots \\ a_{m1}(\underline{x}_1(r), \overline{x}_1(r)) + ... + a_{mn}(\underline{x}_n(r), \overline{x}_n(r)) = (\underline{y}_m(r), \overline{y}_m(r)), \end{cases} \tag{3}$$

where the coefficient matrix$A = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$ is a crisp $m \times n$ matrix and $y_j \epsilon E$, $1 \leq i \leq m$ is called a fuzzy linear system of equations (FLSE)$^{\cdot}$

Definition 3. A fuzzy number vector $X = (x_1, x_2, ..., x_n)^T$given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n, 0 \leq r \leq 1, \tag{4}$$

is called a solution of the fuzzy linear system (3) , if

$$\underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y_i}, \tag{5}$$

$$\overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y_i}. \tag{6}$$

If, for a particular $i, a_{ij} \geq 0$,$1 \leq j \leq n$, we simply get

$$\sum_{j=1}^{n} a_{ij} \underline{x_j} = \underline{y_i}, \tag{7}$$

$$\sum_{j=1}^{n} a_{ij} \overline{x_j} = \overline{y_i}. \tag{8}$$

Consequently, in order to solve the system given by Eq. (3) one must solve a $2m \times 2n$ crisp linear system where the right-hand side column is the function

vector $(\underline{y}_1(r), \underline{y}_2(r), ...,$
$\underline{y}_m(r), \overline{y_1}(r), \overline{y}_2(r), ..., \overline{y}_m(r))^T$. We get the $2m \times 2n$ linear system

$$\begin{cases} s_{11}\underline{x}_1 + ... + s_{1n}\underline{x}_n + s_{1,n+1}\overline{x}_1 + ... + s_{1,2n}\overline{x}_n = \underline{y}_1 \\ \vdots \\ s_{m,1}\underline{x}_1 + ... + s_{m,n}\underline{x}_n + s_{m,n+1}\overline{x}_1 + ... + s_{m,2n}\overline{x}_m = \underline{y}_m \\ s_{m+1,1}\underline{x}_1 + ... + s_{m+1,n}\underline{x}_n + s_{m+1,n+1}\overline{x}_1 + ... + s_{m+1,2n}\overline{x}_n = \overline{y}_1 \\ \vdots \\ s_{2m,1}\underline{x}_1 + ... + s_{2m,n}\underline{x}_n + s_{2m,n+1}\overline{x}_1 + ... + s_{2m,2n}\overline{x}_n = \overline{y}_m \end{cases} \tag{9}$$

where s_{ij} are determined as follows:

$$\begin{cases} a_{ij} \geq 0 \Rightarrow s_{ij} = s_{i+m,j+n} = a_{ij}, \\ a_{ij} \leq 0 \Rightarrow s_{i+m,j} = s_{i,j+n} = a_{ij}, \end{cases} \qquad (1 \leq i \leq m \ , 1 \leq j \leq n). \tag{10}$$

and any s_{ij} which is not determined is zero such that:

$$A = S_1 + S_2. \tag{11}$$

Using matrix notation we get

$$SX = Y, \tag{12}$$

where $S = (s_{ij})$, $1 \leq i \leq 2m$, $1 \leq j \leq 2n$, and

$$X = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ . \\ . \\ . \\ \underline{x}_n \\ \overline{x}_1 \\ \overline{x}_2 \\ . \\ . \\ . \\ \overline{x}_n \end{pmatrix}, Y = \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ . \\ . \\ . \\ \underline{y}_m \\ \overline{y}_1 \\ \overline{y}_2 \\ . \\ . \\ . \\ \overline{y}_m \end{pmatrix}. \tag{13}$$

Therefore,

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X}_1 \\ \overline{X}_2 \end{pmatrix} = \begin{pmatrix} \underline{Y}_1 \\ \overline{Y}_2 \end{pmatrix}. \tag{14}$$

Theorem. The matrix S is a non-singular if and only if the matrix $A = S_1 + S_2$and $S_1 - S_2$ are both non-singular.

Definition 4. Let $X = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n\}$ denote the unique solution of $SX = Y$. The fuzzy number vector $U = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \leq i \leq n\}$ defined by

$$\begin{aligned} \underline{u}_i(r) &= \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\}, \\ \overline{u}_i(r) &= \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\}, \end{aligned} \tag{15}$$

is called the fuzzy solution of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r)), \quad 1 \leq i \leq n$ are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r), \overline{u}_i(r) = \overline{x}_i(r), 1 \leq i \leq n$ and U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

3. Greville's method for solving fuzzy linear systems

Here we first state some important results concerning to the Greville's method for solving linear systems in crisp environment. Then we shall apply this method for solving fuzzy linear systems.

3.1. Greville's method

Greville's method is one of the oldest pseudoinversion methods (see [6]). It can be considered as analogous to the bordering technique in conventional inversion (see [4]).
Let A be an m by n matrix with columns $a_1, a_2, \ldots, a_n$. Greville 's method consists of n steps. At the kth step, the pseudo inverse ${A_k}^+$ is calculated for the matrix

$$A_k = (a_1, \ldots, a_k).$$

The first step of the method is different from the rest. It is described by the formula

$$ {A_1}^+ = {a_1}^+ = \begin{cases} 0 & a_1 = 0, \\ \\ {a_1}^*/({a_1}^* a_1), & a_1 \neq 0. \end{cases} \tag{16}$$

The remaining steps are similar. We describe the kth step, assuming that $k \geq 2$ and the matrix ${A_{k-1}}^+$ has already been found . We write the matrix ${A_k}^+$ calculated at this step in the block form

$${A_k}^+ = \begin{pmatrix} B_k \\ \\ b_k \end{pmatrix}, \tag{17}$$

where b_k is row vector of dimension m. The step starts by calculating the $(k-1)$-dimensional vector

$$d_k = {A_{k-1}}^+ a_k.$$

then, the m-dimensional vector

$$c_k = a_k - A_{k-1} d_k$$

is calculated. If $c_k \neq 0$, we set

$$b_k = {c_k}^+ = (a_k - A_{k-1} d_k)^+ \tag{18}$$

(the pseudo inverse of a nonzero vector is defined by formula (16)). If $c_k = 0$, then

$$b_k = (1 + {d_k}^* d_k)^{-1} {d_k}^* {A_{k-1}}^+. \tag{19}$$

Finally, we calculate the $(k-1)$-by-m matrix

$$B_k = {A_{k-1}}^+ - d_k b_k. \tag{20}$$

Formulas (18), (19) and (20) combined determine the matrix ${A_k}^+$ (see equation (17)). If $n \geq m$, then the number of steps in Greville's method can be reduced based on the well-know formula $(A^T)^+ = (A^+)^T$. In this case, the method is applied to the n-by-m matrix A^T and consists of m steps. The transpose of the matrix calculated by the method is the desired matrix A^+. We return to discussing the efficiency of this technique in next section.

3.2. Greville's method for solving FLSE

We know if A be a non-singular or non-quadratic matrix, then the normal solution of system $Ax = b$ can be expressed

$$x = A^+b$$

. Now again consider the extended crisp linear system concerning to the rectangular fuzzy linear system (3), in the matrix form as follows:

$$Sx = y.$$

We are going to apply the Greville's method for the above system. In this case we computing S^+ using the Greville's method we have:

$$x = S^+y.$$

4. Numerical examples

Here we shall illustrate the proposed method as given in the last section with solving two fuzzy linear systems.

Example 1: Use Greville's algorithm to obtain a solution of the fuzzy linear system:

$x_1 - x_2 = (-7 + 2r, -3 - 2r)$,
$x_1 + 3x_2 = (19 + 4r, 27 - 4r)$.

The extended 4×4 matrix is

$$S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

The generalized inverse of matrix S by Greville's algorithm is:

$$S^+ = \begin{pmatrix} 0.1250 & -0.1250 & -0.3750 & 0.3750 \\ -0.3750 & 0.3750 & 0.1250 & -0.1250 \\ -0.3750 & 0.3750 & 0.1250 & -0.1250 \\ 0.1250 & -0.1250 & -0.3750 & 0.3750 \end{pmatrix}.$$

Now by using $x = S^+b$ we have:

$$x = (1 + r, 6 + r, 3 - r, 8 - r).$$

Example 2: Use Greville's algorithm to obtain a solution of the fuzzy linear system:

$4x_1 + 2x_2 = (4 + 4r, 10 - 2r),$
$-x_1 + 3x_2 = (1 + r, 3 - r),$
$2x_1 + x_2 = (2 + 2r, 5 - r).$

The extended 6×4 matrix is

$$S = \begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix},$$

and then S^+ is computed as:

$$S^+ = \begin{pmatrix} 0.1286 & -0.1714 & 0.1286 & -0.0214 & 0.0286 & 0.0214 \\ -0.0071 & 0.3429 & -0.0071 & 0.0429 & -0.0571 & 0.0429 \\ -0.0214 & 0.0286 & -0.0214 & 0.1286 & -0.1714 & 0.1286 \\ 0.0429 & 0.0571 & 0.0429 & 0.0071 & 0.3429 & -0.0071 \end{pmatrix}.$$

Now using Greville's method, we obtain:

$$x = S^+b = (0.514 + 0.914r, 0.971 + 0.171r, 1.914 - 0.485r, 1.171 - 0.028r).$$

So,

$$\mathbf{x}_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (0.514 + 0.914r, 1.914 - 0.485r),$$
$$\mathbf{x}_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (0.971 + 0.171r, 1.171 - 0.028r).$$

Conclusion

In this paper a rectangular from of fuzzy linear system of equations is considered. With this structure, a fuzzy linear system with a coefficients matrix A is transformed into a $2m \times 2n$ crisp linear system S. The system is then solved with crisp variables and parameters and the solution vector is either a strong fuzzy solution or a weak fuzzy solution. Since in many cases S is singular or in the rectangular form, thus using pesudoinverse matrix will be useful for solving fuzzy linear system of equations. So we proposed the Greville's method for these kinds fuzzy linear systems. Finally we illustrated our method and compared with two common methods such as LU decomposition method (see [1]), QR decomposition method (see [7]) and saw the results of these experimental are equal.

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