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A New Proof of Watson's Theorem for the Series ${}_3F_2(1)$

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Abstract

We give a new proof of the classical Watson theorem for the summation of a ${}_3F_2$ hypergeometric series of unit argument. The proof relies on the two well-known Gauss summation theorems for the ${}_2F_1$ function.

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1. Introduction

The classical Watson theorem for the summation of a ${}_3F_2$ hypergeometric function of unit argument takes the form

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2},\ 2c \end{matrix};1\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\,\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})\,\Gamma(c+\frac{1}{2}-\frac{1}{2}a)\,\Gamma(c+\frac{1}{2}-\frac{1}{2}b)} \tag{1.1} $$

provided $\mathrm{Re}(2c-a-b) > -1$ and the parameters are such that the series on the left is defined. The proof of this result when one of the parameters a or b is a negative integer was given by Watson in [7], and subsequently was established more generally in the non-terminating case by Whipple in [8].

The standard proof of the general case given in [2, p. 149; 6, p. 54] relies on the following transformation due to Thomae

$$
{}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix}; 1\right) = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(s)}{\Gamma(a)\,\Gamma(b+s)\,\Gamma(c+s)}\,{}_3F_2\left(\begin{matrix} d-a,\ e-a,\ s \\ b+s,\ c+s \end{matrix}; 1\right),
$$

where $s = d + e - a - b - c$ is the parametric excess, combined with Dixon's theorem for the evaluation of the sum on the right when $d = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}$ and $e = 2c$. An alternative and more involved proof [4, p. 363] exploits the quadratic transformations for the Gauss hypergeometric function. A third proof, due to Bhatt in [3], exploits a known relation between the F_2 and F_4 Appell functions combined with a comparison of the coefficients in their series expansions.

In this note, we give a simple proof of (1.1) that relies only on the well-known Gauss summation theorems for the ${}_2F_1$ function, namely [1, pp. 556, 557]

$$
{}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \qquad \mathrm{Re}(c-a-b) > 0 \tag{1.2}
$$

and

$$
{}_2F_1(a, b; \tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2}; \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\,\Gamma(\frac{1}{2}b + \frac{1}{2})}. \tag{1.3}
$$

We shall also require the following elementary identities for the Pochhammer symbol, or ascending factorial, $(a)_n = \Gamma(a+n)/\Gamma(a)$ given by

$$
(a)_{2m} = 2^{2m}(\tfrac{1}{2}a)_m\,(\tfrac{1}{2}a + \tfrac{1}{2})_m, \qquad (a)_{k+2m} = (a)_{2m}\,(a+2m)_k \tag{1.4}
$$

for nonnegative integers m and k, together with

Lemma 1 *Let k be a nonnegative integer and c be (in general) a complex parameter satisfying $2c \neq -1, -2, \ldots$. Then*

$$
\frac{(c)_k}{(2c)_k} = \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{2^{-k-2m}k!}{(c+\frac{1}{2})_m m!\,(k-2m)!}, \tag{1.5}
$$

where $\lfloor k/2 \rfloor$ is the integer part of $k/2$.

The proof of this lemma uses (1.2) to express the ratio of Pochhammer symbols as a terminating Gauss hypergeometric function in the form

$$
\begin{aligned}
\frac{(c)_k}{(2c)_k} &= 2^{-k}{}_2F_1(-\tfrac{1}{2}k, \tfrac{1}{2} - \tfrac{1}{2}k; c + \tfrac{1}{2}; 1) \\
&= 2^{-k}\sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(-\frac{1}{2}k)_m\,(\frac{1}{2} - \frac{1}{2}k)_m}{(c+\frac{1}{2})_m\, m!}.
\end{aligned}
$$

The result (1.5) then follows upon making use of the identity

$$(-\tfrac{1}{2}k)_m\,(\tfrac{1}{2}-\tfrac{1}{2}k)_m=\frac{2^{-2m}k!}{(k-2m)!}.$$

2. Proof of Watson's theorem (1.1)

We denote the left-hand side of (1.1) by F and express the ${}_3F_2$ function as a series to find

$$\begin{aligned}F &= \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k\,(c)_k}{(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_k\,(2c)_k\,k!}\\ &= \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_k\,k!}\sum_{m=0}^{\lfloor k/2\rfloor}\frac{2^{-k-2m}k!}{(c+\frac{1}{2})_m\,m!\,(k-2m)!}\end{aligned}$$

by Lemma 1. Upon reversal of the order of summation, making use of the easily established result [5, p. 57]

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\lfloor k/2\rfloor}A(m,k)=\sum_{m=0}^{\infty}\sum_{k=2m}^{\infty}A(m,k)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}A(m,k+2m),$$

then

$$\begin{aligned}F &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a)_{k+2m}\,(b)_{k+2m}\,2^{-k-4m}}{(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_{k+2m}\,(c+\frac{1}{2})_m\,m!\,k!}\\ &= \sum_{m=0}^{\infty}\frac{(a)_{2m}\,(b)_{2m}\,2^{-4m}}{(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_{2m}\,(c+\frac{1}{2})_m\,m!}\sum_{k=0}^{\infty}\frac{(a+2m)_k\,(b+2m)_k\,2^{-k}}{(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2m)_k\,k!}\end{aligned}\qquad(2.1)$$

by the second equation in (1.4).

The inner sum in (2.1) can be expressed as a ${}_2F_1$ function in the form

$${}_2F_1\left(\begin{matrix}a+2m,\ b+2m\\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2m\end{matrix};\frac{1}{2}\right)=\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})}\,\frac{(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_{2m}}{(\frac{1}{2}a+\frac{1}{2})_m\,(\frac{1}{2}b+\frac{1}{2})_m}$$

which has been summed by Gauss' second theorem in (1.3). Substitution of this summation into (2.1), combined with use of the first equation in (1.4), then yields

$$F=\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}a)_m\,(\frac{1}{2}b)_m}{(c+\frac{1}{2})_m\,m!}.$$

This last sum can be summed by Gauss' first theorem in (1.2) when $\mathrm{Re}(2c-a-b)>-1$, and the desired result in (1.1) follows. This completes the proof of Watson's theorem.

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