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Strong Convergence Theorems for Nonself I-Asymptotically Quasi-Nonexpansive Mappings[1]

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Abstract

Let X be a real uniformly convex Banach space, C be a nonempty closed convex subset of X. Let $I : C \to X$ be a nonself asymptotically nonexpansive mapping and $T : C \to X$ be a nonself I-asymptotically quasi-nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by: for any given $x_1 \in C$,

$$\begin{array}{rcl} y_n & = & P(\beta_n T(PT)^{n-1}x_n + (1-\beta_n)x_n), \\ x_{n+1} & = & P(\alpha_n I(PI)^{n-1}y_n + (1-\alpha_n)x_n), \end{array} \qquad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$. The strong convergnece theorems of $\{x_n\}$ to some $x \in F(T) \cap F(I)$ are proved under some appropriate conditions.

Keywords: Common fixed point; nonself I-asymptotically quasi-nonexpansive mapping; Condition (A'); Uniformly convex Banach space

1 Introduction

Let C be a nonempty subset of a real Banach space X. Let T be a self-mapping of C. T is said to be asymptotically nonexpansive if there exists a

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real sequence $\{\lambda_n\} \subset [0, +\infty)$, with $lim_{n\to\infty}\lambda_n = 0$ such that $\|T^n x - T^n y\| \leq (1+\lambda_n)\|x-y\|$ for all $x, y \in C$. T is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. A mapping T is called uniformly L-Lipschitzian if there exists a real number $L > 0$ such that $\|T^n x - T^n y\| \leq L\,\|x - y\|$ for every $x, y \in K$ and each $n \geq 1$. It was proved in [4] that if X is uniformly convex and if C is a bounded closed convex subset of X, then every asymptotically nonexpansive mapping has a fixed point.

Let $T, I : C \to C$, then T is called I-nonexpansive on C if $\|Tu - Tv\| \leq \|Iu - Iv\|$ for all $u, v \in C$. T is called I-asymptotically nonexpansive on C if there exists a sequence $\{\gamma_k\} \subset [0, \infty)$ with $lim_{k\to\infty}\gamma_k = 0$ such that $\|T^k u - T^k v\| \leq (\gamma_k + 1)\|I^k u - I^k v\|$ for all $u, v \in C$ and $k = 1, 2\cdots$. T is called quasi-nonexpansive provided $\|Tu - f\| \leq \|u - f\|$ for all $u \in C$ and $f \in F(T)$ and $k \geq 1$. T is called asymptotically quasi-nonexpansive if there exists a sequence $\{\mu_k\} \subset [0, \infty)$ with $lim_{k\to\infty}\mu_k = 0$ such that $\|T^k u - f\| \leq (\mu_k + 1)\|u - f\|$ for all $u \in C$ and $f \in F(T)$ and $k \geq 1$.

In the past few decades, many results of fixed points on asymptotically nonexpansive, quasi-nonexpansive and asymptotically quasi-nonexpansive mappings in Banach sapce and metric spaces (see, e.g., [3,5,14]) have been obtained. Very recently, Rhoades and Temir [10] studied weak convergence theorems for I-nonexpansive mappings. Temir and Gul [17] also studied the weak convergence theorems for I-asymptotically quasi-nonexpansive mapping in Hilbert space. But they did not obtain any strong convergence theorems for these mappings.

Ishikawa iteration scheme is defined as follows: $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n T y_n, \\ y_n &= (1 - b_n)x_n + b_n T x_n, \end{aligned} \qquad n \geq 1, \tag{1.1}$$

where T is a mapping on C, $\{a_n\}$ and $\{b_n\}$ are two real sequences in [0,1]. The iteration scheme has been widely used to approximate fixed points of nonlinear mappings [see, e.g. 1,7,9,11]. The convexity of C then ensures that the sequence $\{x_n\}$ generated by (1.1) is well defined. But, when C is a proper subset of the real Banach space X and T maps C into X (as the case in many applications), then the sequence given by (1.1) may not be well defined. In recent years, some authors [see, e.g. 6,13,19] obtained the strong convergence theorems for nonself nonexpansive mappings by modifying the iteration (1.1). In 2003, Chidume, Ofoedu and Zegeye [2] introduced the concept of nonself asymptotically nonexpansive mapping and obtained some strong and weak convergence theorems for such mappings by modifying (1.1) as follows: for $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= P((1 - a_n)x_n + a_n T(PT)^{n-1} y_n), \\ y_n &= P((1 - b_n)x_n + b_n T(PT)^{n-1} x_n), \end{aligned} \qquad n \geq 1, \tag{1.2}$$

where P is a retraction from X onto C, $\{a_n\}$ and $\{b_n\}$ are two real sequences in [0,1].

Definition 1.1. Let C be a nonempty subset of a real normed space X, $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for each $n \in \mathbf{N}$,

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \quad for \;\; every \;\; x, y \in C.$$

T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \quad for \;\; every \;\; x, y \in C.$$

Recently, Wang [18] further extended Chidume, Ofoedu and Zegeye's results and obtained the strong and weak convergence theorems of $\{x_n\}$ to common fixed points of a pair of nonself asymptotically nonexpansive mappings.

From Definition 1.1, it is easy to see that every nonself asymptotically nonexpansive mapping is uniformly L-Lipschitzian, where $L = \sup k_n$.

Motivated by above works, we introduce concept of nonself I-asymptotically quasi-nonexpansive mapping to generalize the concept introduced by Chidume, Ofoedu and Zegeye [2]. The strong convergence theorems for such mappings are also obtained.

2 Preliminaries

Throughout this paper, we denote the set of all fixed points of a mapping T by $F(T)$, $T^0 = E$, where E denotes the mapping $E : C \to C$ defined by $Ex = x$, respectively. A subset C of X is said to be retract if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. It is well known that every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to C$ is said to be a retraction if $P^2 = P$. Let D be subsets of C where C is a subset of a Banach space X. Then a mapping $P : C \to D$ is said to be sunny if for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$, $P(Px + t(x - Px)) = Px$. Let C be a subset of a Banach space X. For all $x \in C$, define a set $I_C(x)$ by $I_C(x) = \{x + \lambda(y - x) : \lambda > 0, y \in C\}$. A nonself mapping $T : C \to X$ is said to be inward if $Tx \in I_C(x)$ for all $x \in C$ and T is said to be weakly inward if $Tx \in \overline{I_C(x)}$ for all $x \in C$.

Definition 2.1. Let C be a nonempty closed convex subset of a real uniformly convex Banach space X and $T : C \to X$ be a non-self mapping. Then the mapping T is said to be

(1) demiclosed at y if whenever $\{x_n\} \subset C$ such that $x_n \rightharpoonup x \in C$ and $Tx_n \to y$ then $Tx = y$.

(2) semi-compact if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges strongly to some x^* in K.

(3) completely continuous if the sequence $\{x_n\}$ in C converges weakly to x_0 implies that $\{Tx_n\}$ converges strongly to Tx_0.

For approximating fixed points of nonexpansive mappings, Senter and Dotson [12] introduced a Condition (A). Later on, Maiti and Ghosh [8], Tan and Xu [16] studied the Condition (A) and pointed out that Condition (A) is weaker than the requirement of semi-compactness on mapping.

Definition 2.2. A mapping $T : C \to C$ is said to satisfy Condition (A) if there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in (0, +\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = inf\{d(x, x^*) : x^* \in F(T)\}$, $F(T)$ is the fixed point set of T.

For two mappings, Condition (A) can be written as follow.

Definition 2.3. The mappings $S, T : C \to C$ are said to satisfy Condition (A') if there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, +\infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, F_1))$ for all $x \in C$, where $F_1 = F(S) \bigcap F(T)$ and $d(x, F_1) = inf\{d(x, x^*) : x^* \in F_1\}$.

We restate the following lemmas which play key roles in our proofs.

Lemma 2.1 [14]. Let $\{\alpha_n\}$ and $\{t_n\}$ be two nonnegative sequences satisfying

$$\alpha_{n+1} \leq \alpha_n + t_n, \quad \forall n \geq 1.$$

If $\Sigma_{n=1}^{\infty} t_n < \infty$, then $lim_{n\to\infty} \alpha_n$ exists.

Lemma 2.2 [11]. Let X be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 [2]. Let X be a real uniformly convex Banach space, C be a nonempty closed subset of X, and let T be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $E - T$ is demiclosed at zero.

Lemma 2.4 [15]. Let X be a real smooth Banach space, C be a nonempty closed convex subset of X with P as a sunny nonexpansive retraction and $T : C \to X$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.

3 Main Results

Firstly, we introduce the following definition.

Definition 3.1 Let C be a nonempty closed convex subset of a real Banach space X, $T, I : C \to X$ be two non-self mappings. Then the mapping T is said to be a nonself I-asymptotically quasi-nonexpansive mapping if there exists a sequence $\{l_n\} \subset [1, \infty)$ with $l_n \to 1$ as $n \to \infty$ such that for each $n \in \mathbf{N}$,

$$\|T(PT)^{n-1}x - f\| \leq l_n \|I(PI)^{n-1}x - f\|, \quad for\ all\ x \in C\ and\ f \in F(T) \cap F(I),$$

where P is a retraction from X onto C.

Secondly, for approximating the common fixed points of nonself I-asymptotically quasi-nonexpansive mapping T and nonself mapping I, we generalize the scheme (1.2) as follows: for given $x_1 \in C$,

$$\begin{array}{rcl} x_{n+1} & = & P(\alpha_n I(PI)^{n-1}y_n + (1-\alpha_n)x_n) \\ y_n & = & P(\beta_n T(PT)^{n-1}x_n + (1-\beta_n)x_n) \end{array} \; n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0, 1]$.

Lemma 3.1. Let C be a nonempty closed convex subset of a normed linear space X. Let $T : C \to X$ be a nonself I-asymptotically quasi-nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $I : C \to X$ be a nonself asymptotically nonexpansive mapping with sequence $\{l_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, P be a retraction from X onto C. Suppose the sequence $\{x_n\}$ is generated by (3.1). If $F_1 = F(T) \cap F(I) \neq \emptyset$, then $lim_{n\to\infty}\|x_n - x^*\|$ exists for any $x^* \in F_1$.

Proof. Setting $k_n = 1+s_n, l_n = 1+r_n$. Since $\sum_{n=1}^{\infty}(k_n-1) < \infty, \sum_{n=1}^{\infty}(l_n-1) < \infty$, so $\sum_{n=1}^{\infty} r_n < \infty, \sum_{n=1}^{\infty} s_n < \infty$. Taking $x^* \in F_1$, we have

$$\begin{array}{rl} & \|x_{n+1} - x^*\| \leq \|\alpha_n I(PI)^{n-1}y_n + (1-\alpha_n)x_n - x^*\| \\ \leq & \alpha_n\|I(PI)^{n-1}y_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq & \alpha_n(1+r_n)\|y_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq & \alpha_n(1+r_n)\|\beta_n(T(PT)^{n-1}x_n - x^*) + (1-\beta_n)(x_n - x^*)\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq & \alpha_n\beta_n(1+r_n)(1+s_n)\|I(PI)^{n-1}x_n - x^*\| + \alpha_n(1-\beta_n)(1+r_n)\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq & \alpha_n\beta_n(1+s_n)(1+r_n)^2\|x_n - x^*\| + \alpha_n(1-\beta_n)(1+r_n)^2(1+s_n)\|x_n - x^*\| \\ & +(1-\alpha_n)(1+r_n)^2(1+s_n)\|x_n - x^*\| \\ \leq & (1+r_n)^2(1+s_n)\|x_n - x^*\|. \end{array}$$

Since $\sum_{n=1}^{\infty} r_n < \infty, \sum_{n=1}^{\infty} s_n < \infty$, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. This completes the proof.

Lemma 3.2. Let X be a uniformly convex Banach space. Let $C, T, I, \{x_n\}$ be same as in Lemma 3.1. If T is uniformly L-Lipschitzian for some $L > 0$ and $F_1 \neq \emptyset$, then $lim_{n\to\infty}\|Tx_n - x_n\| = lim_{n\to\infty}\|Ix_n - x_n\| = 0$.

Proof. By Lemma 3.1, for any $x^* \in F_1$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists, then $\{x_n\}$ is bounded. Assume $\lim_{n\to\infty} \|x_n - x^*\| = c \geq 0$. From (3.1), we have

$$\begin{aligned}
\|y_n - x^*\| &\leq \|\beta_n T(PT)^{n-1}x_n + (1-\beta_n)x_n - x^*\| \\
&\leq \beta_n \|T(PT)^{n-1}x_n - x^*\| + (1-\beta_n)\|x_n - x^*\| \\
&\leq \beta_n(1+s_n)\|I(PI)^{n-1}x_n - x^*\| + (1-\beta_n)\|x_n - x^*\| \\
&\leq \beta_n(1+s_n)(1+r_n)\|x_n - x^*\| + (1-\beta_n)(1+s_n)(1+r_n)\|x_n - x^*\| \\
&\leq (1+s_n)(1+r_n)\|x_n - x^*\|.
\end{aligned}$$

Taking limsup on both sides in above inequality, we obtain

$$limsup_{n\to\infty}\|y_n - x^*\| \leq c. \tag{3.2}$$

Since

$$\|I(PI)^{n-1}y_n - x^*\| \leq (1+r_n)\|y_n - x^*\|.$$

Taking limsup on both sides in above inequality and using (3.2) we have

$$limsup_{n\to\infty}\|I(PI)^{n-1}y_n - x^*\| \leq c.$$

Since $lim_{n\to\infty}\|x_{n+1} - x^*\| = c$, then $lim_{n\to\infty}\|\alpha_n(I(PI)^{n-1}y_n - x^*) + (1 - \alpha_n)(x_n - x^*)\| = c$. It follows from Lemma 2.2 that

$$lim_{n\to\infty}\|I(PI)^{n-1}y_n - x_n\| = 0. \tag{3.3}$$

Next,

$$\begin{aligned}
\|x_n - x^*\| &\leq \|x_n - I(PI)^{n-1}y_n\| + \|I(PI)^{n-1}y_n - x^*\| \\
&\leq \|x_n - I(PI)^{n-1}y_n\| + (1+r_n)\|y_n - x^*\|
\end{aligned}$$

gives that $c = lim_{n\to\infty}\|x_n - x^*\| \leq liminf_{n\to\infty}\|y_n - x^*\|$. By (3.2), we have $lim_{n\to\infty}\|y_n - x^*\| = c$.

Since $\|T(PT)^{n-1}x_n - x^*\| \leq (1+s_n)\|I(PI)^{n-1}x_n - x^*\| \leq (1+s_n)(1+r_n)\|x_n - x^*\|$, we have $limsup_{n\to\infty}\|T(PT)^{n-1}x_n - x^*\| \leq c$. Further, $lim_{n\to\infty}\|y_n - x^*\| = c$ means that

$$lim_{n\to\infty}\|\beta_n(T(PT)^{n-1}x_n - x^*) + (1-\beta_n)(x_n - x^*)\| = c.$$

Thus by Lemma 2.2, we have

$$lim_{n\to\infty}\|T(PT)^{n-1}x_n - x_n\| = 0. \tag{3.4}$$

Also,

$$\begin{aligned} & \|I(PI)^{n-1}x_n - x_n\| \\ \leq & \|I(PI)^{n-1}x_n - I(PI)^{n-1}y_n\| + \|I(PI)^{n-1}y_n - x_n\| \\ \leq & (1+r_n)\|x_n - y_n\| + \|I(PI)^{n-1}y_n - x_n\| \\ = & (1+r_n)\|\beta_n(x_n - T(PT)^{n-1}x_n)\| + \|I(PI)^{n-1}y_n - x_n\| \\ \leq & \beta_n(1+r_n)\|x_n - T(PT)^{n-1}x_n\| + \|I(PI)^{n-1}y_n - x_n\|. \end{aligned}$$

Thus by (3.3) and (3.4), we have

$$lim_{n\to\infty}\|I(PI)^{n-1}x_n - x_n\| = 0. \tag{3.5}$$

Further, it follows from (3.4) and (3.5) that

$$lim_{n\to\infty}\|I(PI)^{n-1}x_n - T(PT)^{n-1}x_n\| = 0. \tag{3.6}$$

In addition, $\|x_{n+1} - x_n\| \leq \alpha_n\|I(PI)^{n-1}y_n - x_n\|$, by (3.3)

$$lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Since $\|I(PI)^{n-1}y_n - x_{n+1}\| \leq \|I(PI)^{n-1}y_n - x_n\| + \|x_n - x_{n+1}\|$, we have

$$lim_{n\to\infty}\|I(PI)^{n-1}y_n - x_{n+1}\| = 0. \tag{3.8}$$

So,

$$\begin{aligned} \|x_{n+1} - y_n\| \leq & \|x_{n+1} - I(PI)^{n-1}y_n\| + \|I(PI)^{n-1}y_n - y_n\| \\ \leq & \|x_{n+1} - I(PI)^{n-1}y_n\| + \|y_n - x_n\| + \|x_n - I(PI)^{n-1}y_n\| \\ \leq & \|x_{n+1} - I(PI)^{n-1}y_n\| + \beta_n\|T(PT)^{n-1}x_n - x_n\| + \|x_n - I(PI)^{n-1}y_n\|. \end{aligned}$$

Using (3.3), (3.4) and (3.8), we obtain

$$lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.9}$$

$$\begin{aligned} \|x_n - Ix_n\| = & \|x_n - I(PI)^{n-1}x_n + I(PI)^{n-1}x_n - I(PI)^{n-1}y_{n-1} + I(PI)^{n-1}y_{n-1} - Ix_n\| \\ \leq & \|x_n - I(PI)^{n-1}x_n\| + \|I(PI)^{n-1}x_n - I(PI)^{n-1}y_{n-1}\| + \|I(PI)^{n-1}y_{n-1} - Ix_n\| \\ \leq & \|x_n - I(PI)^{n-1}x_n\| + (1+r_n)\|x_n - y_{n-1}\| + (1+r_1)\|I(PI)^{n-2}y_{n-1} - x_n\|. \end{aligned}$$

It follows from (3.5), (3.8) and (3.9), we obtain

$$lim_{n\to\infty}\|x_n - Ix_n\| = 0. \tag{3.10}$$

Since T is uniformly L-Lipschitzian for some $L > 0$,

$$\begin{aligned} \|Tx_n - x_n\| \leq & \|T(PT)^{n-1}x_n - x_n\| + \|T(PT)^{n-1}x_n - Tx_n\| \\ \leq & \|T(PT)^{n-1}x_n - x_n\| + L\|T(PT)^{n-2}x_n - x_n\| \\ \leq & \|T(PT)^{n-1}x_n - x_n\| + L\|T(PT)^{n-2}x_n - T(PT)^{n-2}x_{n-1}\| \\ & + L\|T(PT)^{n-2}x_{n-1} - x_{n-1}\| + L\|x_{n-1} - x_n\| \\ \leq & \|T(PT)^{n-1}x_n - x_n\| + L^2\|x_n - x_{n-1}\| + (L + L^2)\|T(PT)^{n-2}x_{n-1} - x_{n-1}\|. \end{aligned}$$

By (3.4) and (3.7), we have

$$lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{3.11}$$

This completes the proof.

Theorem 3.3. Let X be a uniformly convex Banach space and $C, T, I, \{x_n\}$ be same as in Lemma 3.2. If T, I satisfy the Condition (A'), then$\{x_n\}$ converges strongly to a common fixed point of T and I.

Proof. By Lemma 3.1, $lim_{n\to\infty}\|x_n - x^*\| = c \geq 0$ exists for all $x^* \in F_1$. If $c = 0$, there is nothing to prove.

Suppose that $c > 0$, by Lemma 3.2, we have

$$lim_{n\to\infty}\|Tx_n - x_n\| = lim_{n\to\infty}\|Ix_n - x_n\| = 0.$$

In the proof of Lemma 3.1, we obtain

$$\|x_{n+1} - x^*\| \leq (1 + r_n)^2(1 + s_n)\|x_n - x^*\|. \tag{3.12}$$

From (3.12), we have $0 \leq d(x_{n+1}, F_1) \leq d(x_n, F_1)$. Then, $lim_{n\to\infty}d(x_n, F_1)$ exists. Now condition (A') guarantees that

$$lim_{n\to\infty}f(d(x_n, F_1)) = 0. \tag{3.13}$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that $lim_{n\to\infty}d(x_n, F_1) = 0$.

For any $\varepsilon > 0$, since $lim_{n\to\infty}d(x_n, F_1) = 0$, there exists natural number N_1 such that when $n \geq N_1$, $d(x_n, F_1) < \frac{\varepsilon}{3}$. Thus, there exists $x'^* \in F_1$ such that for above ε there exists positive integer $N_2 \geq N_1$ such that as $n \geq N_2$, $\|x_n - x'^*\| < \frac{\varepsilon}{2}$. Now for arbitrary $n, m \geq N_2$, consider $\|x_n - x_m\| \leq \|x_n - x'^*\| + \|x_m - x'^*\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This implies $\{x_n\}$ is a cauchy sequence.

Let $x_n \to x$. Now $lim_{n\to\infty}d(x_n, F_1) = 0$ gives that $d(x, F_1) = 0$. By the routine method we can easily show that F_1 is closed, therefore $x \in F_1$. The proof is completed.

Theorem 3.4. Let $X, C, T, I, \{x_n\}$ be same as in Lemma 3.2. If $F_1 \neq \emptyset$ and I is a semi-compact mapping, then $\{x_n\}$ converges to a common fixed point of T and I.

Proof. Since I is a semi-compact mapping, $\{x_n\}$ is bounded and $lim_{n\to\infty}\|x_n - Ix_n\| = 0$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges to x^*. It follows from Lemma 2.3, $x^* \in F(I)$. Further, $\|Tx_{n_j} - x^*\| \leq \|Tx_{n_j} - x_{n_j}\| + \|x_{n_j} - Ix_{n_j}\| + \|Ix_{n_j} - Ix^*\| + \|Ix^* - x^*\|$. Thus, $lim_{n\to\infty}\|Tx_{n_j} - x^*\| = 0$. This implies that $\{Tx_{n_j}\}$ converges strongly to x^*. Since T is uniformly L-Lipschitzian for some $L > 0$, $Tx^* = x^*$. Since the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges to x^* and $lim_{n\to\infty}\|x_n - x^*\|$ exists, then $\{x_n\}$ converges strongly to the common fixed point $x^* \in F(T) \cap F(I)$. The proof is completed.

Theorem 3.5. Let $X, C, T, I, \{x_n\}$ be same as in Lemma 3.2. If $F_1 \neq \emptyset$, I is completely continuous mapping, then $\{x_n\}$ converges strongly to a common fixed point of T and I.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded. Since $lim_{n\to\infty}\|x_n - Ix_n\| = lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $\{Tx_n\}$ and $\{Ix_n\}$ are bounded. Since I is completely continuous, that exists subsequence $\{Ix_{n_j}\}$ of $\{Ix_n\}$ such that $\{Ix_{n_j}\} \to p$ as $j \to \infty$. Thus, we have $lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = lim_{j\to\infty}\|x_{n_j} - Ix_{n_j}\| = 0$. So, by the continuity of I and Lemma 2.3, we have $lim_{j\to\infty}\|x_{n_j} - p\| = 0$, $p \in F(I)$. And $\|Tx_{n_j} - p\| \leq \|Tx_{n_j} - x_{n_j}\| + \|x_{n_j} - Ix_{n_j}\| + \|Ix_{n_j} - Ip\| + \|Ip - p\|$. Thus, $lim_{n\to\infty}\|Tx_{n_j} - p\| = 0$. This implies that $\{Tx_{n_j}\}$ converges strongly to p. Since T is continuous, $Tp = p$. Further, from Lemma 3.1, $lim_{n\to\infty}\|x_n - p\|$ exists. Thus $lim_{n\to\infty}\|x_n - p\| = 0$. The proof is completed.

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