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Iterative Method for Equilibrium Problems and Fixed Point Problem for Countable Nonexpansive Mappings in Hilbert Spaces

Issara Inchan

Department of Mathematics and Computer
Uttaradit Rajabhat University
Uttaradit 53000, Thailand
peissara@uru.ac.th

Abstract

In this paper, we introduce the iterative schemes by the iterative method for finding a common element of the set of an equilibrium problem and the set of fixed points of countable nonexpansive mapping in a Hilbert space. These result extended and improved the corresponding result of Plubtieng and Panpaeng [A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 336 (2007) 455-469], and many others.

Keywords: countable nonexpansive mappings, equilibrium problems, minimization problem

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all} \quad y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the innitial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \ \text{ for all } \ x \in H. \tag{2}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{3}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2003, Xu [10] prove that the sequence $\{x_n\}$ defined by iterative method below, with the initial guess $x_0 \in H$, chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \tag{4}$$

converges strongy to the unique solution of the minimization problem (3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions that will be made prccise in section 3.

On the other hand, Moudafi [4] introduced the viscosity approximation method for nonexpansive mappings (see [11] for further develovements in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary innitial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geq 0, \tag{5}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [4, 11] that under certain appropriate condition imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (5) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in C. \tag{6}$$

Recently, Marino and Xu [5] was combine the iterative method (4) with the viscosity approximation (5) and consider the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{7}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (7) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C. \tag{8}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{9}$$

where h is a potential function for γf (i.e., $h^{'}(x) = \gamma f(x)$ for $x \in H$).

In 2007, S. Takahashi and W. Takahashi [9] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1) and the set of fixed points of a nonexpansive mapping in Hilbert space. Let $S : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbf{N}. \end{cases} \tag{10}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)}f(z)$.

Moreover, S. Plubtieng and R. Punpaeng [7] introduced an iterative scheme by the general iterative method for finding a common element of the set of solution (1) and the set of fixed points of a nonexpansive mapping in Hilbert space.

Let $S : H \to H$ be a nonexpansive mapping. starting with an arbitrary $x_1 \in H$, define sequence $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Su_n, \quad \forall n \in \mathbf{N}. \end{cases} \tag{11}$$

They proved that if the sequence $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generate by (11) converges strongly to the unique solution of variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F(S) \cap EP(F), \tag{12}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{13}$$

where h is a potential function for γf.

In this paper, motivated S. Takahashi and W. Takahashi [9] and S. Plubtieng and R. Punpaeng [7], we introduce an iterative scheme by the general iterative method for finding a common element of the set of solution (1) and the set of fixed points of a nonexpansive mapping in Hilbert space. Let $\{T_n\}_{n=1}^{\infty}$ be family of nonexpansive mapping on H, starting with an arbitrary $x_1 \in H$, define sequence $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_n u_n, \quad \forall n \in \mathbf{N}. \end{cases} \tag{14}$$

We will prove in section 3 that if the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generate by (14) converges strongly to the unique solution of variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F(S) \cap EP(F), \tag{15}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{16}$$

where h is a potential function for γf.

2 Preliminary Notes

In this section, we collect some lemmas which will be used in the proof for the main result in next section.

Lemma 2.1 *[1] Let X be a real uniformly smooth Banach space and let $J : X \longrightarrow 2^{X^*}$ be the normalized duality mapping. Then for any $x, y \in X$ we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j\rangle, \quad \forall j \in J(x + y).$$

Lemma 2.2 *[11] Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,*

$$a_{n+1} \leq (1 - \gamma_n)a_n + b_n, \ n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$, and $\{b_n\}$ is a sequence in $\mathbf{R}$ such that:
i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;
ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 *[1] Suppose that $\Sigma_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty$. Then, for each $y \in C$, converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by*

$$Ty = \lim_{n\to\infty} T_n y \text{ for all } y \in C.$$

Then $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$.

Lemma 2.4 *[8] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5 *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x||^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.6 *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.7 *[5] Let H be a Hilbert space, C be a nonempty closed convex subset of H, and $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$,*

$$\langle x - y, (A - \gamma f)x - A(A - \gamma f)y\rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.

Lemma 2.8 *[5] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y \in C$,

$$\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 2.9 *[2] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in C.$$

The following lemma was also given in [3].

Lemma 2.10 *[3] Assume that $F : C \times C \to \mathbf{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

1. T_r is single-valued;

2. T_r is firmly nonexpansive, i. e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for all $x, y \in H$;

3. $F(T_r) = EP(F)$;

4. $EP(F)$ is closed and convex.

3 Main Results

In this section, we prove strong convergence theorems of sequence generate by (14) for countable nonexpansive mappings in Hilbert spaces.

Lemma 3.1 *[7] Let H be a real Hilbert space. Let F be a bifunction from $H \times H \to \mathbf{R}$ satisfying $(A1) - (A4)$ and let S be a nonexpansive mapping on H such that $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $\alpha \in (0, 1)$, and let A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ and $\{u_n\}$ be sequence generated by $x_1 \in C$ and*

$$\begin{cases} x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Su_n, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \end{cases} \tag{17}$$

for all $n \in \mathbf{N}$, where $u_n = T_{r_n}x_n, \{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0, \liminf_{n\to\infty} \gamma_n > 0$. Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to z, where $z = P_{F(T)\cap EP(F)}(I - A + \gamma f)$ is the unique solution of the variational inequalities (12).

Theorem 3.2 *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be a family of nonexpansive mappings on H with $\Sigma_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$, such that the common fixed point set $F(T) = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let F be a bifunction from $H \times H \to \mathbf{R}$ satisfying $(A1) - (A4)$ and $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $\alpha \in (0, 1)$, and let A a strongly* *positive bounded linear operator with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequence generated by* $x_1 \in H$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_n u_n, \end{cases} \tag{18}$$

for all $n \in \mathbf{N}$, *where* $u_n = T_{r_n}x_n, \{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty}\beta_n = 0$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, $\liminf_{n\to\infty}\gamma_n > 0$ *and* $\Sigma_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$. *Then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* z, *where* $z = P_{F(T)\cap EP(F)}(I - A + \gamma f)$ *is the unique solution of the variational inequalities (12).*

Proof. Note that from the condition $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H, then

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I - \alpha_n A)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0,\end{aligned}$$

that is to say $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle((1-\beta_n)I - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned}$$

We now observe that $\{x_n\}$ is bounded. Indeed pick any $p \in F(T)\cap EP(F)$. Then from $u_n = T_{r_n}x_n$, we have $\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|$ for all $n \in \mathbf{N}$. Thus, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_n u_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(T_n u_n - p)\| \\ &\leq \|\alpha_n\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| + \|((1-\beta_n)I - \alpha_n A)\|\|T_n u_n - p\| \\ &\leq \|\alpha_n\gamma f(x_n) - \gamma f(p) + \gamma f(p) - Ap\| + \beta_n\|x_n - p\| \\ &\quad + (1 - \beta_n - \alpha_n\overline{\gamma})\|\|u_n - p\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| \\ &\quad + (1 - \beta_n - \alpha_n\overline{\gamma})\|\|T_{r_n}x_n - p\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| \\ &\quad + (1 - \beta_n - \alpha_n\overline{\gamma})\|\|x_n - p\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n(\overline{\gamma} - \gamma\alpha)\tfrac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}.\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}\}, n \geq 0,$$

and hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{f(x_n)\}$ and $\{T_nx_n\}$ are bounded. Next, we show that $\|x_{n+1}-x_n\| \to 0$. We have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)T_nu_n-(\alpha_{n-1}\gamma f(x_{n-1})\\
&\quad +\beta_{n-1}x_{n-1}+((1-\beta_{n-1})I-\alpha_{n-1}A)T_{n-1}u_{n-1})\|\\
&\le \|\alpha_n\gamma f(x_n)-\alpha_{n-1}\gamma f(x_{n-1})\|+\|((1-\beta_n)I-\alpha_nA)T_nu_n\\
&\quad -((1-\beta_{n-1})I-\alpha_{n-1}A)T_{n-1}u_{n-1})\|+\|\beta_nx_n-\beta_{n-1}x_{n-1}\|\\
&\le \|\alpha_n\gamma f(x_n)-\alpha_n\gamma f(x_{n-1})+\alpha_n\gamma f(x_{n-1})\alpha_{n-1}\gamma f(x_{n-1})\|\\
&\quad +\|\beta_nx_n-\beta_nx_{n-1}+\beta_nx_{n-1}-\beta_{n-1}x_{n-1}\|\\
&\quad +\|((1-\beta_n)I-\alpha_nA)T_nu_n-((1-\beta_{n-1})I-\alpha_{n-1}A)T_nu_n)\\
&\quad +((1-\beta_{n-1})I-\alpha_{n-1}A)T_nu_n)-((1-\beta_{n-1})I-\alpha_{n-1}A)T_{n-1}u_{n-1})\|\\
&\le \alpha_n\gamma\alpha\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+\beta_n\|x_n-x_{n-1}\|\\
&\quad +|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\|(1-\beta_n)I-\alpha_nA\|\|T_nu_n-T_nu_{n-1}\|\\
&\quad +\|(1-\beta_n)I-\alpha_nA)-((1-\beta_{n-1})I-\alpha_{n-1}A)\|\|T_nu_{n-1}\|\\
&\quad +\|(1-\beta_{n-1})I-\alpha_{n-1}A\|\|T_nu_{n-1}-T_{n-1}u_{n-1}\|\\
&\le \alpha_n\gamma\alpha\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|K+\beta_n\|x_n-x_{n-1}\|\\
&\quad +|\beta_n-\beta_{n-1}|K+|\beta_{n-1}-\beta_n|K+(1-\beta_n-\alpha_n\overline{\gamma})\|u_n-u_{n-1}\|\\
&\quad +|\alpha_{n-1}-\alpha_n|K+(1-\beta_{n-1}-\alpha_{n-1}\overline{\gamma})\|T_nu_{n-1}-T_{n-1}u_{n-1}\|
\end{aligned} \tag{19}$$

where

$$K=\sup\{\|f(x_n)\|+\|x_n\|+\|T_nu_n\|+\|AT_nu_n\| : n\in \mathbf{N}\}<\infty.$$

On the other hand, we note that

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0,\ \forall y\in H, \tag{20}$$

and

$$F(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle \ge 0,\ \forall y\in H. \tag{21}$$

Putting $y=u_{n+1}$ in (20) and $y=u_n$ in (21), we have

$F(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle \ge 0$ and

$F(u_{n+1},u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle \ge 0$

and hence

$\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\ge 0.$

Since $\liminf_{n\to\infty} r_n>0$, we assume that there exists a real number b such that $r_n>b>0$ for all $n\in\mathbf{N}$. Thus, we have

$$\begin{aligned}
\|u_{n+1}-u_n\|^2 &\le \langle u_{n+1}-u_n,x_{n+1}-x_n+(1-\tfrac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\rangle\\
&\le \|u_{n+1}-u_n\|\{\|x_{n+1}-x_n\|+|1-\tfrac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\|\}
\end{aligned}$$

and hence

$$\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\| + |1-\tfrac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\|$$
$$\le \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|L \qquad (22)$$

where $L=\sup\{\|u_{n+1}-x_{n+1}\| : n\in \mathbf{N}\}$.
From (19) we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \alpha_n\gamma\alpha\|x_n-x_{n-1}\| + \gamma\|\alpha_n-\alpha_{n-1}|K\\ &\quad +\beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|K + |\beta_{n-1}-\beta_n|K\\ &\quad +|\alpha_{n-1}-\alpha_n|K + (1-\beta_{n-1}-\alpha_{n-1}\overline{\gamma})\|T_nu_{n-1}-T_{n-1}u_{n-1}\|\\ &\quad +(1-\beta_n-\alpha_n\overline{\gamma})\{\|x_n-x_{n-1}\| + \tfrac{1}{b}|r_n-r_{n-1}|\}.\\ &= (1-\alpha_n\overline{\gamma}+\alpha_n\gamma\alpha)\|x_n-x_{n-1}\| + ((1+\gamma)|\alpha_n-\alpha_{n-1}|\\ &\quad +2|\beta_n-\beta_{n-1}|)K + (1-\beta_{n-1}-\alpha_{n-1}\overline{\gamma})\|T_nu_{n-1}-T_{n-1}u_{n-1}\|\\ &\quad +\tfrac{1-\beta_n-\alpha_n\overline{\gamma}}{b}|r_n-r_{n-1}|L\\ &\le (1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| + ((1+\gamma)|\alpha_n-\alpha_{n-1}|+2|\beta_n-\beta_{n-1}|)K\\ &\quad +(1-\beta_{n-1}-\alpha_{n-1}\overline{\gamma})\sup\{\|T_{n+1}z-T_nz\| : z\in B\} + \tfrac{L}{b}|r_n-r_{n-1}|\\ &= (1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-x_{n-1}\| + b_n\end{aligned}$$

where $b_n := ((1+\gamma)|\alpha_n-\alpha_{n-1}|+2|\beta_n-\beta_{n-1}|)K+(1-\beta_{n-1}-\alpha_{n-1}\overline{\gamma})\sup\{\|T_{n+1}z-T_nz\| : z\in B\} + \frac{L}{b}|r_n-r_{n-1}|$ hence $\Sigma_{n=1}^{\infty}|b_n|<\infty$. By assumptions we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From (22) and $|r_n-r_{n-1}|\to 0$, we have $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$.
It follows that

$$\begin{aligned}\|x_n-T_nu_n\| &\le \|x_n-T_{n-1}u_{n-1}\| + \|T_{n-1}u_{n-1}-T_nu_n\|\\ &= \|\alpha_{n-1}\gamma f(x_{n-1}) + \beta_{n-1}x_{n-1} + ((1-\beta_{n-1})I-\alpha_{n-1}A)T_{n-1}u_{n-1}\\ &\quad -T_{n-1}u_{n-1}\| + \|u_{n-1}-u_n\|\\ &\le \alpha_n\|\gamma f(x_{n-1}) - AT_{n-1}u_{n-1}\| + \beta_{n-1}\|x_{n-1}-T_{n-1}u_{n-1}\|\\ &\quad +\|u_{n-1}-u_n\|.\end{aligned}$$

From $\lim_{n\to\infty}\alpha_n=0, \lim_{n\to\infty}\beta_n=0$, we have $\lim_{n\to\infty}\|x_n-T_nu_n\|=0$. For $p\in F(T)\cap EP(F)$, we have

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}x_n-T_{r_n}p\|^2\\ &\le \langle T_{r_n}x_n-T_{r_n}p, x_n-p\rangle\\ &= \langle u_n-p, x_n-p\rangle = \tfrac{1}{2}(\|u_n-p\|^2+\|x_n-p\|^2-\|x_n-u_n\|^2)\end{aligned}$$

and hence

$$\|u_n-p\|^2 \le \|x_n-p\|^2 - \|x_n-u_n\|^2. \qquad (23)$$

Therefore, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_nx_n + ((1-\beta_n)I-\alpha_nA)T_nu_n-p\|^2\\ &= \|((1-\beta_n)I-\alpha_nA)(T_nu_n-p)+\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)\|^2\\ &\le \|(1-\beta_n)I-\alpha_nA\|^2\|u_n-p\|^2 + 2\alpha_n\langle\gamma f(x_n)-Ap, x_{n+1}-p\rangle\\ &\quad +2\beta_n\langle x_n-p, x_{n+1}-p\rangle\\ &\le (1-\beta_n-\alpha_n\overline{\gamma})^2\|u_n-p\|^2 + 2\alpha_n\langle\gamma f(x_n)-\gamma f(p), x_{n+1}-p\rangle\\ &\quad +2\alpha_n\langle\gamma f(p)-Ap, x_{n+1}-p\rangle + 2\beta_n\|x_n-p\|\|x_{n+1}-p\|\\ &\le (1-\beta_n-\alpha_n\overline{\gamma})^2\|u_n-p\|^2 + 2\alpha_n\alpha\gamma\|x_n-p\|\|x_{n+1}-p\|\end{aligned}$$

$$
\begin{aligned}
&+2\alpha_n\|\gamma f(p) - Ap\|\|x_{n+1} - p\| + 2\beta_n\|x_n - p\|\|x_{n+1} - p\| \\
&\leq (1 - \beta_n - \alpha_n\overline{\gamma})^2\{\|x_n - p\|^2 - \|x_n - u_n\|^2\} \\
&+2\alpha_n\alpha\gamma\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\|\gamma f(p) - Ap\|\|x_{n+1} - p\| \\
&+2\beta_n\|x_n - p\|\|x_{n+1} - p\| \\
&\leq (1 - 2\alpha_n\overline{\gamma} + (\alpha_n\overline{\gamma})^2)\|x_n - p\|^2 - (1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2\} \\
&+2\alpha_n\alpha\gamma\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\|\gamma f(p) - Ap\|\|x_{n+1} - p\| \\
&+2\beta_n\|x_n - p\|\|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 + \alpha_n\overline{\gamma}^2\|x_n - p\|^2 - (1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2\} \\
&+2\alpha_n\alpha\gamma\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\|\gamma f(p) - Ap\|\|x_{n+1} - p\| \\
&+2\beta_n\|x_n - p\|\|x_{n+1} - p\|
\end{aligned}
$$

and hence

$$
\begin{aligned}
(1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\overline{\gamma}^2\|x_n - p\|^2 \\
&+2\alpha_n\alpha\gamma\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\|\gamma f(p) - Ap\|\|x_{n+1} - p\| \\
&+2\beta_n\|x_n - p\|\|x_{n+1} - p\| \\
&\leq \|x_n - x_{n+1}\|\{\|x_n - p\| + \|x_{n+1} - p\|\} + \alpha_n\overline{\gamma}^2\|x_n - p\|^2 \\
&+2\alpha_n\alpha\gamma\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\|\gamma f(p) - Ap\|\|x_{n+1} - p\| \\
&+2\beta_n\|x_n - p\|\|x_{n+1} - p\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0, \lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n = 0$, we have $\|x_n - u_n\| \to 0$. Note that

$$\|T_n u_n - u_n\| \leq \|T_n u_n - x_n\| + \|x_n - u_n\|$$

it follows that $\|T_n u_n - u_n\| \to 0$. Since $\|Tu_n - u_n\| \leq \|Tu_n - T_n u_n\| + \|T_n u_n - x_n\| + \|x_n - u_n\|$ and Lemma 2.3, it follows that $\|Tu_n - u_n\| \to 0$. Next, we show that

$$\limsup_{n\to\infty}\langle(A - \gamma f)z, z - x_n\rangle \leq 0, \tag{24}$$

where $z \in F(T) \cap EP(F)$ is a unique solution of the variational inequality (12). We can choose $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty}\langle(A - \gamma f)z, z - x_{n_i}\rangle = \limsup_{n\to\infty}\langle(A - \gamma f)z, z - x_n\rangle. \tag{25}$$

Since $\{u_{n_i}\}$ is bounded, without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. We show that $w \in EP(F)$. It follows by (18) and $(A2)$ that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, it follows by $(A4)$ that $0 \geq F(y, w)$ for all $y \in H$. For $t \in (0, 1]$ and $y \in H$ let $y_t = ty + (1 - t)w$. Since $y, w \in H$, we have $y_t \in H$ and hence $F(y_t, w) \leq 0$. So from $(A1)$ and $(A4)$ we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, w)$$

$$\leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From $(A3)$, we have $0 \leq F(w, y)$ for all $y \in H$ and hence $w \in EP(F)$.

We shall show that $w \in F(T)$. Assume that $w \neq F(T)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Tw$, it follows by Opial's condition (see [6]) that

$$\begin{aligned}\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Tw\| \\ &\leq \liminf_{i\to\infty}(\|u_{n_i} - Tu_{n_i}\| + \|Tu_{n_i} - Tw\|) \\ &\leq \liminf_{i\to\infty} \|u_{n_i} - w\|.\end{aligned}$$

A contradiction. So, we get $w \in F(T)$. Therefore $w \in F(T) \cap EP(F)$. From Lemma 3.1, we have $z \in F(T) \cap EP(F)$ is unique solution of variational inequality (12), it follows that

$$\limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle = \lim_{i\to\infty}\langle (A-\gamma f)z, z-x_{n_i}\rangle = \langle (A-\gamma f)z, z-w\rangle \leq 0. \tag{26}$$

From $\|T_n u_n - x_n\| \to 0$ as $n \to \infty$ and equation (26), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle (A - \gamma f)z, z - T_n u_n\rangle &= \limsup_{n\to\infty}\langle (A - \gamma f)z, z - x_n\rangle \\ &+ \limsup_{n\to\infty}\langle (A - \gamma f)z, x_n - T_n u_n\rangle \leq 0.\end{aligned} \tag{27}$$

Finally we prove that $x_n \to z$ as $n \to \infty$. Since and bounded of $\{x_n\}, \{u_n\}$ we set

$$M \geq \|\gamma f(x_n) - z\|^2 + \|T_n u_n - z\|\|\gamma f(x_n) - Az\|$$

From $\|u_n - z\| = \|T_{r_n}x_n - T_{r_n}z\| \leq \|x_n - z\|$, it follows that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_n u_n - z\|^2 \\ &= \|((1-\beta_n)I - \alpha_n A)(T_n u_n - z) + \beta_n(x_n - z) + \alpha_n(\gamma f(x_n) - z)\|^2 \\ &= \|((1-\beta_n)I - \alpha_n A)(T_n u_n - z) + \beta_n(x_n - z)\|^2 + \alpha_n^2\|\gamma f(x_n) - z\|^2 \\ &\quad +2\langle((1-\beta_n)I - \alpha_n A)(T_n u_n - z) + \beta_n(x_n - z), \alpha_n(\gamma f(x_n) - z)\rangle \\ &\leq [(1-\beta_n - \alpha_n\overline{\gamma})\|u_n - z\| + \beta_n\|x_n - z\|]^2 + \alpha_n^2 M \\ &\quad +2\beta_n\alpha_n\langle x_n - z, \gamma f(x_n) - Az\rangle + 2(1-\beta_n)\alpha_n\langle T_n u_n - z, \gamma f(x_n) - Az\rangle \\ &\quad -2\alpha_n^2\langle A(T_n u_n - z), \gamma f(x_n) - Az\rangle \\ &\leq [(1-\beta_n-\alpha_n\overline{\gamma})\|x_n - z\| + \beta_n\|x_n - z\|]^2 + 2\beta_n\alpha_n\langle x_n - z, \gamma f(x_n) - \gamma f(z)\rangle \\ &\quad +\alpha_n^2 M + 2(1-\beta_n)\alpha_n\langle T_n u_n - z, \gamma f(x_n) - \gamma f(z)\rangle \\ &\quad +2\beta_n\alpha_n\langle x_n - z, \gamma f(z) - Az\rangle \\ &\quad +2(1-\beta_n)\alpha_n\langle T_n u_n - z, \gamma f(z) - Az\rangle - 2\alpha_n^2\langle A(T_n u_n - z), \gamma f(x_n) - Az\rangle \\ &\leq (1-\alpha_n\overline{\gamma})^2\|x_n - z\|^2 + 2\beta_n\alpha_n\gamma\alpha\|x_n - z\|^2 \\ &\quad +2\beta_n\alpha_n\langle x_n - z, \gamma f(z) - Az\rangle + \alpha_n^2 M \\ &\quad +2(1-\beta_n)\alpha_n\|T_n u_n - z\|\|\gamma f(x_n) - \gamma f(z)\| \\ &\quad +2(1-\beta_n)\alpha_n\langle T_n u_n - z, \gamma f(z) - Az\rangle + 2\alpha_n^2 M \\ &\leq (1-\alpha_n\overline{\gamma})^2\|x_n - z\|^2 + 2\beta_n\alpha_n\gamma\alpha\|x_n - z\|^2 \\ &\quad +2(1-\beta_n)\alpha_n\gamma\alpha\|x_n - z\|^2 + 3\alpha_n^2 M + 2\beta_n\alpha_n\langle x_n - z, \gamma f(z) - Az\rangle \\ &\quad +2(1-\beta_n)\alpha_n\langle T_n u_n - z, \gamma f(z) - Az\rangle \\ &\leq [1 - 2(\overline{\gamma} - \alpha\gamma)\alpha_n]\|x_n - z\|^2 + \alpha_n(2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle\end{aligned}$$

$$+2(1-\beta_n)\langle T_n u_n - z, \gamma f(z) - Az\rangle + 3\alpha_n M).$$
$$=: (1-\gamma_n)\|x_n - z\|^2 + \delta_n$$

where $\gamma_n = 2(\overline{\gamma} - \alpha\gamma)\alpha_n$ and $\delta_n = \alpha_n(2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle T_n u_n - z, \gamma f(z) - Az\rangle + 3\alpha_n M)$. From $\Sigma_{n=1}^{\infty}\alpha_n = \infty$, we have $\Sigma_{n=1}^{\infty}\gamma_n = \infty$ and (35), (36), we have $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$. Hence, by Lemma 2.2, the sequence $\{x_n\}$ converge strongly to z. ⋄

If $\beta_n \equiv 0$ and $T_n \equiv S$ for all $n \in \mathbf{N}$, in Theorem 3.2 we obtain the following corollary.

Corollary 3.3 *[7] Let H be a real Hilbert space. Let F be a bifunction from $H \times H \to \mathbf{R}$ satisfying (A1) − (A4) and let S be a nonexpansive mapping on H such that $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$, and let A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ and $\{u_n\}$ be sequence generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Su_n, \end{cases}$$

for all $n \in \mathbf{N}$, where $u_n = T_{r_n}x_n, \{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\liminf_{n\to\infty}\gamma_n > 0$ and $\Sigma_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to z, where $z = P_{F(T)\cap EP(F)}(I - A + \gamma f)$ is the unique solution of the variational inequalities (12). ⋄

Theorem 3.4 *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be a family of nonexpansive mappings on H with $\Sigma_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$, such that the common fixed point set $F(T) = \cap_{n=1}^{\infty}F(T_n) \neq \emptyset$. Let F be a bifunction from $H \times H \to \mathbf{R}$ satisfying (A1) − (A4) and $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$, and let A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ and $\{u_n\}$ be sequence generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_n u_n, \end{cases} \tag{28}$$

for all $n \in \mathbf{N}$, where $u_n = T_{r_n}x_n, \{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, $\liminf_{n\to\infty}\gamma_n > 0$ and $\Sigma_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to z, where $z = P_{F(T)\cap EP(F)}(I - A + \gamma f)$ is the unique solution of the variational inequalities (12).

Proof. In the proof of theorem 3.2 we have, $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{f(x_n)\}$ and $\{T_nx_n\}$ are bounded. Next, we show that $\|x_{n+1}-x_n\| \to 0$. Define the sequence $z_n = \frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)T_nu_n}{1-\beta_n}$, such that $x_{n+1}=\beta_nx_n+(1-\beta_n)z_n$, $n\geq 0$. Observe that from the definition of z_n we obtain

$$\begin{aligned}
z_{n+1}-z_n &= \tfrac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)T_{n+1}u_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)T_nu_n}{1-\beta_n}\\
&= \tfrac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} - \tfrac{\alpha_n\gamma f(x_{n+1})}{1-\beta_{n+1}} + \tfrac{\alpha_n\gamma f(x_{n+1})}{1-\beta_{n+1}} - \tfrac{\alpha_n\gamma f(x_n)}{1-\beta_{n+1}} + \tfrac{\alpha_n\gamma f(x_n)}{1-\beta_{n+1}}\\
&\quad - \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)T_{n+1}u_{n+1}}{1-\beta_{n+1}} - \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)T_{n+1}u_n}{1-\beta_{n+1}}\\
&\quad + \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)T_{n+1}u_n}{1-\beta_{n+1}} - \tfrac{((1-\beta_n)I-\alpha_nA)T_{n+1}u_n}{1-\beta_{n+1}} + \tfrac{((1-\beta_n)I-\alpha_nA)T_{n+1}u_n}{1-\beta_{n+1}}\\
&\quad - \tfrac{\alpha_n\gamma f(x_n)}{1-\beta_n} - \tfrac{((1-\beta_n)I-\alpha_nA)T_nu_n}{1-\beta_n} - \tfrac{((1-\beta_n)I-\alpha_nA)T_nu_n}{1-\beta_{n+1}} + \tfrac{((1-\beta_n)I-\alpha_nA)T_nu_n}{1-\beta_{n+1}}\\
&= (\alpha_{n+1}-\alpha_n)\tfrac{(\gamma f(x_{n+1}))}{\beta_{n+1}} + \tfrac{\alpha_n\gamma}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))\\
&\quad +(\alpha_n\gamma f(x_n))(\tfrac{1}{1-\beta_{n+1}} - \tfrac{1}{1-\beta_n}) + \tfrac{((1-\beta_{n+1})I-\alpha_{n+1}A)(T_{n+1}u_{n+1}-T_{n+1}u_n)}{1-\beta_{n+1}}\\
&\quad + \tfrac{[((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_nA)]T_{n+1}u_n}{1-\beta_{n+1}} + \tfrac{((1-\beta_n)I-\alpha_nA)(T_{n+1}u_n-T_nu_n)}{1-\beta_{n+1}}\\
&\quad +((1-\beta_n)I-\alpha_nA)T_nu_n(\tfrac{1}{1-\beta_{n+1}} - \tfrac{1}{1-\beta_n}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|z_{n+1}-z_n\| &\leq |\alpha_{n+1}-\alpha_n|\|\tfrac{(\gamma f(x_{n+1}))}{1-\beta_{n+1}}\| + \tfrac{\alpha_n\gamma}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|\\
&\quad +\alpha_n\gamma\|f(x_n)\||\tfrac{1}{1-\beta_{n+1}} - \tfrac{1}{1-\beta_n}| + \tfrac{\|(1-\beta_{n+1})I-\alpha_{n+1}A\|}{1-\beta_{n+1}}\|T_{n+1}u_{n+1}-T_{n+1}u_n\|\\
&\quad +\tfrac{\|(1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_nA)\|\|T_{n+1}u_n\|}{1-\beta_{n+1}} + \tfrac{\|((1-\beta_n)I-\alpha_nA)\|\|T_{n+1}u_n-T_nu_n\|}{1-\beta_{n+1}}\\
&\quad +\|((1-\beta_n)I-\alpha_nA)T_nu_n\||\tfrac{1}{1-\beta_{n+1}} - \tfrac{1}{1-\beta_n}|.\\
&\leq |\alpha_{n+1}-\alpha_n|\|\tfrac{(\gamma f(x_{n+1}))}{\beta_{n+1}}\| + \tfrac{\alpha_n\alpha\gamma}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \alpha_n\gamma\|f(x_n)\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}|\\
&\quad +\tfrac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}\|u_{n+1}-u_n\| + \tfrac{[|\beta_n-\beta_{n+1}|+|\alpha_n-\alpha_{n+1}|\|A\|]\|T_{n+1}u_n\|}{1-\beta_{n+1}}\\
&\quad +\tfrac{\|(1-\beta_n-\alpha_n\overline{\gamma})\|\|T_{n+1}u_n-T_nu_n\|}{1-\beta_{n+1}} + \|(1-\beta_n-\alpha_n\overline{\gamma})\|T_nu_n\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}|.
\end{aligned}$$

From equation 22 in Theorem 3.2, we have

$$\|u_{n+1}-u_n\| \leq \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|L \tag{29}$$

it follows that

$$\begin{aligned}
\|z_{n+1}-z_n\| &\leq |\alpha_{n+1}-\alpha_n|\|\tfrac{(\gamma f(x_{n+1}))}{\beta_{n+1}}\| + \tfrac{\alpha_n\alpha\gamma}{1-\beta_{n+1}}\|x_{n+1}-x_n\|\\
&\quad +\alpha_n\gamma\|f(x_n)\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}| + \tfrac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}(\|x_{n+1}-x_n\|\\
&\quad +\tfrac{1}{b}|r_{n+1}-r_n|L) + \tfrac{[|\beta_n-\beta_{n+1}|+|\alpha_n-\alpha_{n+1}|\|A\|]\|T_{n+1}u_n\|}{1-\beta_{n+1}}\\
&\quad +\tfrac{\|(1-\beta_n-\alpha_n\overline{\gamma})\|\|T_{n+1}u_n-T_nu_n\|}{1-\beta_{n+1}} + \|(1-\beta_n-\alpha_n\overline{\gamma})\|T_nu_n\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}|\\
&\leq |\alpha_{n+1}-\alpha_n|\|\tfrac{(\gamma f(x_{n+1}))}{\beta_{n+1}}\| + \tfrac{\alpha_n\alpha\gamma}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \alpha_n\gamma\|f(x_n)\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}|\\
&\quad +(\|x_{n+1}-x_n\| + \tfrac{1}{b}|r_{n+1}-r_n|L) + \tfrac{[|\beta_n-\beta_{n+1}|+|\alpha_n-\alpha_{n+1}|\|A\|]\|T_{n+1}u_n\|}{1-\beta_{n+1}}\\
&\quad +\tfrac{\|(1-\beta_n-\alpha_n\overline{\gamma})\|\|T_{n+1}u_n-T_nu_n\|}{1-\beta_{n+1}} + \|(1-\beta_n-\alpha_n\overline{\gamma})\|T_nu_n\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}|
\end{aligned}$$

it implies that

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \leq |\alpha_{n+1}-\alpha_n|\|\tfrac{(\gamma f(x_{n+1}))}{\beta_{n+1}}\| + \tfrac{\alpha_n\alpha\gamma}{1-\beta_{n+1}}\|x_{n+1}-x_n\|$$

$$+\alpha_n\gamma\|f(x_n)\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}| + \tfrac{\|(1-\beta_n-\alpha_n\overline{\gamma})\|\|T_{n+1}u_n-T_nu_n\|}{1-\beta_{n+1}}$$
$$+\tfrac{1}{b}|r_{n+1}-r_n|L + \tfrac{[|\beta_n-\beta_{n+1}|+|\alpha_n-\alpha_{n+1}|\|A\|]\|T_{n+1}u_n\|}{1-\beta_{n+1}}$$
$$+\|(1-\beta_n-\alpha_n\overline{\gamma})\|T_nu_n\||\tfrac{\beta_{n+1}-\beta_n}{(1-\beta_{n+1})(1-\beta_n)}|.$$

Since $\limsup_{n\to\infty}\|T_{n+1}u_n - T_nu_n\| = 0, \lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}|\alpha_{n+1}-\alpha_n| = 0, \lim_{n\to\infty}|\beta_{n+1}-\beta_n| = 0$ and $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$, it follows that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0. \tag{30}$$

From $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ and (30) by Lemma 2.4, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{31}$$

We consider

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(1-\beta_n)z_n - \beta_nx_n - x_n\|\\ &= (1-\beta_n)\|z_n - x_n\|\end{aligned}$$

then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0$$

and from (29), we have $\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0$. Next, we show that $\|x_n - u_n\| \to 0$ as $n\to\infty$. For $p \in F(T)\cap EP(F)$ and from Lemma 2.1, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_nx_n + ((1-\beta_n)I - \alpha_nA)T_nu_n - p\|^2\\ &= \|\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(T_nu_n-p)+\alpha_n(\gamma f(x_n)-Ap\|^2\\ &\leq \|\beta_n(x_n-p) + ((1-\beta_n)I - \alpha_nA)(T_nu_n-p)\|^2\\ &\quad +2\alpha_n\langle\gamma f(x_n)-p, x_{n+1}-p\rangle\\ &= \beta_n\|x_n-p\|^2 + (1-\beta_n)\|T_nu_n-p\|^2 - \beta_n(1-\beta_n)\|T_nu_n-x_n\|^2\\ &\quad +2\alpha_n\langle\gamma f(x_n)-p, x_{n+1}-p\rangle\\ &\leq \beta_n\|x_n-p\|^2 + (1-\beta_n)\|u_n-p\|^2 - \beta_n(1-\beta_n)\|T_nu_n-x_n\|^2\\ &\quad +2\alpha_n\|f(x_n)-p\|\|x_{n+1}-p\|\\ &\leq \beta_n\|x_n-p\|^2 + (1-\beta_n)(\|x_n-p\|^2 - \|x_n-u_n\|^2)\\ &\quad -\beta_n(1-\beta_n)\|T_nu_n-x_n\|^2 + 2\alpha_nK\end{aligned}$$

where $M = \|f(x_n)-p\|\|x_{n+1}-p\|$. Then,

$$\begin{aligned}(1-\beta_n)\|x_n-u_n\|^2 &\leq \beta_n\|x_n-p\|^2 + (1-\beta_n)\|x_n-p\|^2 - \|x_{n+1}-p\|^2\\ &\quad -\beta_n(1-\beta_n)\|T_nu_n-x_n\|^2 + 2\alpha_nK\\ &\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\alpha_nK\\ &= (\|x_n-p\| - \|x_{n+1}-p\|)(\|x_n-p\| + \|x_{n+1}-p\|) + 2\alpha_nK\\ &\leq \|x_n - x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + 2\alpha_nK\\ &\quad \to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence $\lim_{n\to\infty}\|x_n-u_n\| = 0$. Next, we show that $\lim_{n\to\infty}\|T_nu_n - x_n\| = 0$. Since

$$\begin{aligned}\|T_nu_n - x_n\| &\leq \|T_nu_n - x_{n+1}\| + \|x_{n+1}-x_n\|\\ &= \|\alpha_n\gamma f(x_n) + \beta_nx_n + ((1-\beta_n)I - \alpha_nA)T_nu_n + T_nu_n\|\\ &\quad +\|x_{n+1}-x_n\|\\ &\leq \alpha_n\|\gamma f(x_n) - AT_nu_n\| + \beta_n\|x_n - T_nu_n\| + \|x_{n+1}-x_n\|\end{aligned}$$

it follows that

$(1-\beta_n)\|T_nu_n - x_n\| \le \alpha_n\|\gamma f(x_n) - AT_nu_n\| + \|x_{n+1} - x_n\|.$

Since $\{x_n\}$ and $\{u_n\}$ are bounded, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, thus

$$\lim_{n\to\infty}\|T_nu_n - x_n\| = 0. \tag{32}$$

Since $\|T_nu_n - u_n\| \le \|T_nu_n - x_n\| + \|x_n - u_n\|$, it follows that

$$\lim_{n\to\infty}\|T_nu_n - u_n\| = 0. \tag{33}$$

Since $\|Tu_n - u_n\| \le \|Tu_n - T_nu_n\| + \|T_nu_n - u_n\|$, from (32) and Lemma 2.3, we have

$$\lim_{n\to\infty}\|Tu_n - u_n\| = 0. \tag{34}$$

By argument in the proved of Theorem 3.2, we have

$$\limsup_{n\to\infty}\langle (A-\gamma f)z, z - x_n\rangle \le 0, \qquad \text{for all } z \in F(T)\cap EP(F) \tag{35}$$

and

$$\limsup_{n\to\infty}\langle (A-\gamma f)z, z - T_nu_n\rangle \le 0 \qquad \text{for all } z \in F(T)\cap EP(F). \tag{36}$$

Finally we prove that $x_n \to z$ as $n\to\infty$. From Theorem 3.2 we can reduced that

$\|x_n - z\|^2 \le (1-\gamma_n)\|x_n - z\|^2 + \delta_n$

where $\gamma_n = 2(\overline{\gamma}-\alpha\gamma)\alpha_n$ and $\delta_n = \alpha_n(2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle T_nu_n - z, \gamma f(z) - Az\rangle + 3\alpha_n M)$. From $\Sigma_{n=1}^{\infty}\alpha_n = \infty$, we have $\Sigma_{n=1}^{\infty}\gamma_n = \infty$ and (35), (36), we have $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0$. By Lemma 2.2, we have $\lim_{n\to\infty}\|x_n - z\|^2 = 0$. This completed the proof. ◇

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