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A Class of Alternating Group Iterative Method for Diffusion Equations

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Abstract

In this paper, a high order implicit scheme for solving diffusion equations is presented, based on which a class of alternating group explicit iterative method(AGI) is derived. Also the convergence analysis and the stability analysis are given. In the end, the results of numerical experiment are presented, which shows the AGI method is convergent and suitable for parallel computation.

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1 Introduction

we consider the following time-dependent periodic initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le t \le T \\ u(x,0) = f(x), \\ u(x,t) = u(x+1,t). \end{cases} \tag{1.1}$$

Researches on the periodic initial boundary value problem for the diffusion equations have been scarcely presented so far. Considering the stability and accuracy of explicit schemes and the computation difficulty of implicit schemes, it is a important work to construct numerical methods with absolute stability and easy to compute. With the development of parallel computer, researches on parallel numerical methods are getting more and more popular. D. J. Evans and A. R. B. Abdullah presented a class of alternating group method (AGE) for diffusion equation by the specifical combination of several asymmetric schemes in [1], and applied the method to convection-diffusion equation in [2]. The AGE method is widely cared for it is simple for computing, unconditionally stable, and has the property of parallelism. Based on the AGE method, many alternating group methods have been presented such as in [3-6]. Most of the methods inherit the advantages of the AGE method, and are of higher accuracy than the AGE method. But almost all the methods have $O(h^2)$ accuracy for spatial step. On the other hand, researches on alternating group iterative methods are scarcely presented.

The construction of this paper is as follows: In section 2, we present an $O(\tau^2 + h^4)$ order symmetry implicit scheme for solving (1.1) at first. Based on the scheme a class of alternating group explicit iterative method(AGI) is constructed. In section 3, convergence analysis and stability analysis are given. In section 4, results of numerical experiments are presented.

2 The Alternating Group Iterative Method(AGI)

The domain $\Omega : (0,1) \times (0,T)$ will be divided into $(m \times N)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{N}$. Grid points are denoted by (x_i, t_n) or by (i, n), $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1.1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. We present an implicit scheme for solving (1.1) as below:

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{a}{2}[(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2h^2} - \frac{u_{i+2}^{n+1} - 4u_{i+1}^{n+1} + 6u_i^{n+1} - 4u_{i-1}^{n+1} + u_{i-2}^{n+1}}{12h^2}) +$$

$$(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2h^2} - \frac{u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n}{12h^2})]$$

Applying Taylor formula to the scheme at $(x_i, t_{n+\frac{1}{2}})$, we can easily have that the truncation error of the scheme is $O(\tau^2 + h^4)$.

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$, r=$\frac{a\tau}{24h^2}$, then we have:

$$ru_{i-2}^{n+1} - 16ru_{i-1}^{n+1} + (1 + 30r)u_i^{n+1} - 16ru_{i+1}^{n+1} + ru_{i+2}^{n+1} =$$

$$-ru_{i-2}^n + 16ru_{i-1}^n + (1 - 30r)u_i^n + 16ru_{i+1}^n - ru_{i+2}^n \tag{2.1}$$

We denote (2.1) as $AU^{n+1} = F^n$. here $F^n = (2I - A)U^n$.

$$A=\begin{pmatrix} 1+30r & -16r & r & & & & r & -16r \\ -16r & 1+30r & -16r & r & & & & r \\ r & -16r & 1+30r & -16r & r & & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & r & -16r & 1+30r & & -16r & r \\ r & & & r & -16r & & 1+30r & -16r \\ -16r & r & & & r & & -16r & 1+30r \end{pmatrix}_{m\times m}$$

The alternating group iterative method will be constructed in two conditions as follows:
(1)$m=8k,\ k$ is an integer. Let $A=\frac{1}{2}(G_1+G_2)$, here

$$G_1=\begin{pmatrix} G_{11} & & & & \\ & \dots & & & \\ & & \dots & & \\ & & & \dots & \\ & & & & G_{11} \end{pmatrix}_{m\times m},\ G_2=\begin{pmatrix} G_{21} & & & & \overline{G} \\ & G_{11} & & & \\ & & \dots & & \\ & & & G_{11} & \\ \overline{G}^T & & & & G_{21} \end{pmatrix}_{m\times m}$$

$$\overline{G}=\begin{pmatrix} 0 & 0 & 2r & -32r \\ 0 & 0 & 0 & 2r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ G_{21}=\begin{pmatrix} 1+30r & -16r & r & 0 \\ -16r & 1+30r & -16r & r \\ r & -16r & 1+30r & -16r \\ 0 & r & -16r & 1+30r \end{pmatrix}$$

$$G_{11}=\begin{pmatrix} 1+30r & -16r & r & & & & & \\ -16r & 1+30r & -16r & r & & & & \\ r & -16r & 1+30r & -16r & 2r & & & \\ & r & -16r & 1+30r & -32r & 2r & & \\ & & 2r & -32r & 1+30r & -16r & r & \\ & & & 2r & -16r & 1+30r & -16r & r \\ & & & & r & -16r & 1+30r & -16r \\ & & & & & r & -16r & 1+30r \end{pmatrix}$$

Then the alternating group iterative method I can be derived as below:

$$\begin{cases} (\rho I+G_1)\tilde{U}^{n+1}_{k+\frac{1}{2}}=(\rho I-G_2)U^{n+1}_{k}+2F^n \\ (\rho I+G_2)U^{n+1}_{k+1}=(\rho I-G_1)\tilde{U}^{n+1}_{k+\frac{1}{2}}+2F^n \end{cases} \tag{2.2}$$

Here k is the iterative parameter. Thus computing in the whole domain can be splitted into many sub-domains, and can be worked out with several

parallel computers. So the method has the obvious property of parallelism.
(2)$m=8k+4,\ k$ is an integer. Let $A=\frac{1}{2}(\widetilde{G}_1+\widetilde{G}_2)$.

$$\widetilde{G}_1=\begin{pmatrix} G_{11} & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & & & G_{11} & \\ & & & & G_{21} \end{pmatrix}_{m\times m},\ \widetilde{G}_2=\begin{pmatrix} G_{21} & & & & \overline{G} \\ & G_{11} & & & \\ & & \cdots & & \\ & & & \cdots & \\ \widetilde{\overline{G}} & & & & G_{11} \end{pmatrix}_{m\times m},\ \overline{\overline{G}}=\begin{pmatrix} O \\ \overline{G}^T \end{pmatrix}_{8\times 4}$$

Then the alternating group iterative method II can be derived as below:

$$\begin{cases} (\rho I+\widetilde{G}_1)\tilde{U}^{n+1}_{k+\frac{1}{2}}=(\rho I-\widetilde{G}_2)U^{n+1}_{k}+2F^n \\ (\rho I+\widetilde{G}_2)U^{n+1}_{k+1}=(\rho I-\widetilde{G}_1)\tilde{U}^{n+1}_{k+\frac{1}{2}}+2F^n \end{cases} \tag{2.3}$$

3 Convergence And Stability Analysis

In order to verify the convergence of the AGI method, from [7] we have the following lemmas:

Lemma 1 Let $\theta>0$, and $G+G^T$ is nonnegative, then $(\theta I+G)^{-1}$exists, and

$$\|(\theta I+G)^{-1}\|_2\le\theta^{-1} \tag{3.1}$$

Lemma 2 On the conditions of Lemma 1, we have:

$$\|(\theta I-G)(\theta I+G)^{-1}\|_2\le 1 \tag{3.2}$$

Theorem 1 The alternating group iterative method I (2.2) is convergent.

proof: Let $\widehat{G}_1=\frac{1}{r}(G_1-I),\ \widehat{G}_2=\frac{1}{r}(G_2-I)$, then $G_1=I+r\widehat{G}_1,\ G_2=I+r\widehat{G}_2$.

From the construction of the matrixes $\widehat{G}_1$ and $\widehat{G}_2$ we can see they are both nonnegative definite real matrixes. Thus $G_1,\ G_2,\ (G_1+G_1^T),\ (G_2+G_2^T)$ are all nonnegative matrixes. Then we have $\|(\rho I-G_1)(\rho I+G_1)^{-1}\|_2\le 1,\ \|(\rho I-G_2)(\rho I+G_2)^{-1}\|_2\le 1$.

From (2.2), we can obtain $U^{n+1}=GU^n+2(\rho I+G_2)^{-1}[(\rho I-G_1)(\rho I+G_1)^{-1}F^n+F^n],\ G=(\rho I+G_2)^{-1}(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)$ is growth matrix.

Let $\overline{\overline{G}}=(\rho I+G_2)G(\rho I+G_2)^{-1}=(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)(\rho I+G_2)^{-1}$, then $\rho(G)=\rho(\overline{\overline{G}})\le\|\overline{\overline{G}}\|_2\le 1.$, which shows the alternating group method I given by (2.2) is convergent.

Analogously we have:

Theorem 2 The alternating group iterative method II given by (2.3) is also convergent.

We will use the Fourier method to analyze the stability of (2.1). Let $u_i^n = V^n e^{i\alpha x_j}$, then from (2.1) we have

$$V^{n+1} = V^n \frac{1 - 30r - 2rcos(2\alpha h) + 32rcos(\alpha h)}{1 + 30r + 2rcos(2\alpha h) - 32rcos(\alpha h)}. \quad (3.3)$$

Let $p = 30r + 2rcos(2\alpha h) - 32rcos(\alpha h)$, then

$$V^{n+1} = \frac{1-p}{1+p} V^n = EV^n$$

Considering $p = 4r(cos(\alpha h)-1)(cos(\alpha h)+7) \geq 0$, we have $|E| = |\frac{1-p}{1+p}| =\leq 1$, we can get the following theorem:

Theorem 3 The scheme (2.1) is unconditionally stable.

4 Numerical Experiments

We consider the following time-dependent periodic initial boundary value problem of diffusion equations:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq t \leq T \\ u(x,0) = sin(2\pi x), \\ u(x,t) = u(x+1,t). \end{cases} \quad (4.1)$$

The exact solution for the problem is $u(x,t) = e^{-4\pi^2 t} sin(2\pi x)$. Let A.E denote absolute error, while P.E denote relevant error. A.E$=|u_i^n - u(x_i,t_n)|$, P.E$=100 \times |u_i^n - u(x_i,t_n)/u(x_i,t_n)|$. We will use the iterative error 1×10^{-6} to control the process of iterativeness.

Table 1: The numerical results for the iterative scheme I

	$m=16, \tau=10^{-3}, t=100\tau$	$m=16, \tau=10^{-4}, t=100\tau$
A.E	3.801 $\times 10^{-7}$	6.478 $\times 10^{-5}$
P.E	2.132 $\times 10^{-3}$	9.622 $\times 10^{-3}$
average iterative times	2.93	1.46

Table 2: The numerical results for the iterative scheme I

	$m=16, \tau=10^{-4}, t=1000\tau$	$m=24, \tau=10^{-3}, t=100\tau$
A.E	1.769 $\times 10^{-5}$	6.140 $\times 10^{-6}$
P.E	9.169 $\times 10^{-2}$	3.179 $\times 10^{-2}$
average iterative times	1.131	4.56

Table 3: The numerical results for the iterative scheme II

	$m = 20, \tau = 10^{-3}, t = 100\tau$	$m = 20, \tau = 10^{-4}, t = 100\tau$
A.E	1.932×10^{-6}	1.400×10^{-5}
P.E	1.018×10^{-2}	1.932×10^{-2}
average iterative times	3.8	1.11

Table 4: The numerical results for the iterative scheme II

	$m = 20, \tau = 10^{-4}, t = 1000\tau$	$m = 28, \tau = 10^{-3}, t = 100\tau$
A.E	2.950×10^{-6}	8.328×10^{-6}
P.E	2.365×10^{-2}	4.469×10^{-2}
average iterative times	1.022	5.1

From the results of numerical experiments we can see that the numerical solution for the AGI method (2.2) and (2.3) can converge to the exact solution fast, and we can get higher accuracy when the spatial step diminishes, which accords to the conclusion of convergence and error analysis.

References

[1] D. J. Evans, A. R. Abdullah, Group Explicit Method for Parabolic Equations [J]. Inter. J. Compu. Math. 14 (1983) 73-105.

[2] D. J. Evans, A. R. Abdullah, A New Explicit Method for Diffusion-Convection Equation, Comp. Math. Appl. 11 (1985)145-154.

[3] J. Gao, G. He, An unconditionally stable parallel difference scheme for parabolic equations, Appl. Math. Comput. 135 (2003) 391-398.

[4] R. Tavakoli, P. Davami, New stable group explicit finite difference method for solution of diffusion equation, Appl. Math. Comput. 181 (2006) 1379-1386.

[5] S. Zhu, G. Yuan, W. Sun, Convergence and stability of explicit/implicit schemes for parabolic equations with discontinuous coeffients, Int. J. Numer. Anal. Model. 1 (2004) 131-145.

[6] D. J. Evans, H. Bulut, The numerical solution of the telegraph equation by the alternating group explicit(AGE) method[J], Inter. J. of Comp. Math. 80 (2003)1289-1297.

[7] K. R. B. , An alternating Direction Method for Operator Equations[J]. J. S. Indust. Appl. Math(SIAM), 12 (1964) 848-854.

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