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Periodic Boundary Value Problem for Second Order Ordinary Differential Equations

Slimane Benaicha

Department of Mathematics
University of Oran, Es-senia, Algeria
slimanebenaicha@yahoo.fr

Abstract

In this paper we study the periodic boundary value problem for a nonlinear ordinary differential equation of second order. We give sufficient and necessary conditions for the existence of periodic solutions.

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1 Introduction

In this paper we consider the following periodic boundary value problem (PBVP for short) of second order:

$$x^{//} + f(x)\left|x^{/}\right| + g(x) = h(t) \tag{1.1}$$

$$x(0) = x(2\pi), x^{/}(0) = x^{/}(2\pi) \tag{1.2}$$

in which the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ and $h : [0, 2\pi] \longrightarrow \mathbb{R}$ are continuous.The equation (1.1) is called the Lienard equation with forcing term $f(x)$.If f is constant, we obtain the problem studied in [7] .The important case $f = 0$ (also known as the conservative case) is treated in [4],[5] , [8] and [9] .

We shall establish some necessary and sufficient conditions to ensure the existence of a periodic solution for the problem (1.1)-(1.2) following the spirit of [7] .While studying this problem,we distinguish two cases for the nonlinearity namely

(1) when g is decreasing, in this case we use the method of upper and lower solutions,

(2) when g is increasing, first we show that the method of upper and lower solutions is not useful and in this case we employ an abstract existence theorem [3] .

2 Preliminary results

We consider the equation

$$-x^{//} = k(t, x, x^{/}) \tag{2.1}$$

where the function $k : [0, 2\pi] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous.A function $\alpha \in C^2 [0, 2\pi]$ will be called a lower solution of (2.1) on $[0, 2\pi]$ if

$$-\alpha^{//} \leq k(t, \alpha, \alpha^{/})$$

Similary $\beta \in C^2 [0, 2\pi]$ will be called an upper solution of (2.1) on $[0, 2\pi]$ if

$$-\beta^{//} \geq k(t, \beta, \beta^{/})$$

Theorem 2.1[7]: Assume the following

(a) there exist α and β lower and upper solutions of (2.1) on $[0, 2\pi]$,respectively,with $\alpha(0) = \alpha(2\pi)$, $\beta(0) = \beta(2\pi)$,and $\alpha(t) \leq \beta(t)$, $t \in [0, 2\pi]$;

(b) the inequalities $\alpha^{/}(0) \geq \alpha^{/}(2\pi)$ and $\beta^{/}(0) \leq \beta^{/}(2\pi)$ hold;

(c) k satisfies the following Nagumo condition relative to α, β :There exists $e \in C\ [[0, \infty[\, , (0, \infty)]$ such that$|\ k(t, u, v)| \leq e(|v|)$ whenever $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in [0, 2\pi]$ and e is such that

$$\int_0^\infty \frac{sds}{e(s)} = \infty.$$

Then the PBVP

$$-x^{//} = k(t, x, x^{/}), x(0) = x(2\pi), x^{/}(0) = x^{/}(2\pi) \tag{2.2}$$

has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in [0, 2\pi]$.

We write $k(t, x, x^{/}) = f(x) \left|x^{/}\right| + g(x) - h(t)$ and we have the following result.

Lemma 2.1: Let α, β be lower and upper solutions of (1.1),respectively,satisfying the Conditions (a) and (b) of Theorem2.1.Then k satisfies the Nagumo Condition (c) of Theorem 2.1 relative to α, β .

Proof: From the notation of Theorem2.1 we have $k(t, u, v) = f(u) |v| + g(u) - h(t)$.Taking into account that f, g and h are continuous we see that there exist positive constants K, L such that $|\ f(u)| \leq K$ and $|\ g(u)| + |\ h(t| \leq L$, for $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0, 2\pi]$ .Thus we can take $e(s) = Ks + L$.

3 Existence of solutions

Theorem 3.1: We assume that g is decreasing on $\mathbb{R}$ and f satisfies the following condition

(a) $\exists M > 0 : f(x) - f(y) \geq -M(x-y), \forall x, y \in \mathbb{R}.$

Then the PBVP(1.1)-(1.2) has a solution if and only if

$$\omega = \frac{1}{2\pi}\int_0^{2\pi} h(t)dt \in Range\ g.$$

.

Proof: Assume that $\omega \in Range\ g$.Then there exists $r \in \mathbb{R}$ such that $g(r) = \omega$.The problem

$$x^{//} + f(x) \left| x^{/} \right| = h(t) - g(r), \quad x(0) = x(2\pi), x^{/}(0) = x^{/}(2\pi) \tag{3.1}$$

has a solution since $\int_0^{2\pi} [h(t) - g(r)]\, dt = 0$.Let u be the solution of (3.1) satisfying $\int_0^{2\pi} u(t)dt = 0$.Choose constants $a \in \mathbb{R}_-^*$ and $b \in \mathbb{R}_+^*$ such that

$$u(t) + a \leq r \leq u(t) + b, \forall t \in [0, 2\pi]$$

Take $\alpha = u + a$ and $\beta = u + b$.Thus,

$$-\alpha^{//} = -u^{//} = f(u)\left|u^{/}\right| - h(t) + g(r) = f(\alpha - a)\left|\alpha^{/}\right| - h(t) + g(r)$$

By Condition (a) of Theorem 3.1 we obtain for $x = \alpha$ and $y = \alpha - a$

$$\exists M > 0 : f(\alpha) - f(\alpha - a) \geq -Ma$$

since $\alpha^{/} = u^{/}$ then $f(\alpha)\left|\alpha^{/}\right| + M\ a\ \alpha^{/} \geq f(\alpha - a)\left|\alpha^{/}\right|$ and hence

$$-\alpha^{//} \leq f(\alpha)\left|\alpha^{/}\right| + Ma\left|\alpha^{/}\right| - h(t) + g(r) \leq f(\alpha)\left|\alpha^{/}\right| - h(t) + g(\alpha)$$

so that α is lower solution of (1.1) such that

$$\alpha(0) = \alpha(2\pi), \alpha^{/}(0) = \alpha^{/}(2\pi)$$

.

Similary

$$-\beta^{//} = -u^{//} = f(u)\left|u^{/}\right| - h(t) + g(r) = f(\beta - b)\left|\beta^{/}\right| - h(t) + g(r)$$

Again by Condition (a) of Theorem3.1 we obtain for $x = \beta - b$ and $y = \beta$

$$\exists M > 0 : f(\beta - b) - f(\beta) \geq Mb$$

Thus

$$-\beta^{//} \geq f(\beta)\left|\beta^{/}\right| + Mb\left|\beta^{/}\right| - h(t) + g(r) \geq f(\beta)\left|\beta^{/}\right| - h(t) + g(\beta)$$

$$\beta(0) = u(0) + b = u(2\pi) + b = \beta(2\pi)$$

$$\beta^{/}(0) = u^{/}(0) = u^{/}(2\pi) = \beta^{/}(2\pi)$$

so that β is upper solution of (1.1). From Lemma2.1 and Theorem2.1 it follows that the PBVP(1.1)-(1.2) has a solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in [0, 2\pi]$.

Conversely suppose that x is a solution of the PBVP (1.1)-(1.2).Integrating(1.1) on $[0, 2\pi]$ and using (1.2),we get

$$\int_0^{2\pi} g(x(t))dt = \int_0^{2\pi} h(t)dt = 2\pi\omega.$$

Since g is decreasing, we have $g(\infty) \leq g(x) \leq g(-\infty), \forall\, x \in \mathbb{R}$,where

$$g(\infty) = \lim_{x \longrightarrow \infty} g(x),\ g(-\infty) = \lim_{x \longrightarrow -\infty} g(x).$$

Then,we have

$$\omega = \frac{1}{2\pi}\int_0^{2\pi} h(t)dt \in [g(\infty),\ g(-\infty)] = \overline{Range\ g}.$$

Now if $\omega \notin Range\ g$,then either $\omega = g(\infty)$ or $\omega = g(-\infty)$.For $\omega = g(\infty)$ we have $g(x(t)) > \omega, \forall t \in [0, 2\pi]$ and hence $\int_0^{2\pi} g(x(t))dt > 2\pi\omega$,which is a contradiction.For $\omega = g(-\infty)$ we have $g(x(t)) < \omega, \forall t \in [0, 2\pi]$ and hence $\int_0^{2\pi} g(x(t))dt < 2\pi\omega$,which is a contradiction.Then $\omega \in Range\ g$.

Remark : For the case when g is increasing on $\mathbb{R}$, we first show that the method of upper and lower solutions described inTheorem2.1 is not useful since in this case we have $\alpha(t) = \beta(t)$ on $[0, 2\pi]$ which amounts to assuming the existence of a periodic solution.Indeed, assume that Conditions (a) and (b) of Theorem 2.1 hold and also assume that g is strictly increasing.We have from the definition of α and β

$$\beta^{//} - \alpha^{//} \leq f(\alpha)\left|\alpha^{/}\right| - f(\beta)\left|\beta^{/}\right| + g(\alpha) - g(\beta)$$

Integration on $[0, 2\pi]$ together with (1.2) yields

$$\int_0^{2\pi} [g(\alpha(t) - g(\beta(t)]\, dt \geq 0$$

Since g is strictly increasing and $\alpha(t) \leq \beta(t)$ on $[0, 2\pi]$ then

$$\int_0^{2\pi} [g(\alpha(t) - g(\beta(t))] \, dt \leq 0$$

Thus,

$$\int_0^{2\pi} [g(\alpha(t) - g(\beta(t))] \, dt = 0$$

which implies that $\alpha(t) = \beta(t)$ on $[0, 2\pi]$.

Also, we see that the conclusion of Theorem2.1 does not hold in general as may be seen from the following example.

Example: We consider the problem

$$x^{//} + x = h(t) \ , t \in [0, 2\pi]$$

$$x(0) = x(2\pi), x^{/}(0) = x^{/}(2\pi)$$

in this case we have $f(x) = 0$ and $g(x) = x$,so that g is strictly increasing and *Range* $g = \mathbb{R}$. However, this problem has no solution unless

$$\int_0^{2\pi} h(t) \sin t dt = \int_0^{2\pi} h(t) \cos t dt = 0$$

Therefore, in the case when g is increasing,we employ an abstract existence theorem [3] to study the PBVP (1.1)-(1.2). We consider the operator equation

$$Lx = Nx \tag{3.2}$$

in which $L : D(L) \subset E \longmapsto F$ and $N : E \longmapsto F$ are linear and nonlinear operators respectively, and E and F are Banach spaces.

(I) Let us suppose that there exist projection operators $P : E \longmapsto E$ and $Q : F \longmapsto F$ (that is,linear,bounded,and idempotent) such that $E = E_0 \oplus E_1, F = F_0 \oplus F_1$ with $E_0 = PE = \ker L$, $E_1 = (I - P)E$, $F_0 = QF$, $F_1 = RangeL = (I - Q)F$ and $\dim E_0 = \dim F_0 < \infty$ and there exists a linear operator $H : (I - Q)F \longmapsto (I - P)E$,called the partial inverse of L, such that

$$(a) H(I - Q)Lx = (I - P)x, \text{for every } x \in D(L)$$

$$(b) \ QLx = LPx, \ \text{for every } x \in D(L)$$

$$(c)\ LH(I-Q)Nx=(I-Q)Nx, \text{ for every } \ x\in E$$

Equation (3.2) is equivalent to the system of auxiliary and bifurcation equations

$$x=Px+H(I-Q)Nx \tag{3.4}$$

$$Q(Lx-Nx)=0 \tag{3.5}$$

(II) In addition,suppose that there exist continuous maps $B: E\times F\longmapsto \mathbb{R}$,and $J: F_0\longmapsto E_0$ such that

(i) B is bilinear and J is one-to-one and onto;

(ii) $y_0\in F_0,\ y_0=0$ iff $B(\ x_0,\ y_0\ )=0\ \ \forall x_0\in E_0$;

(iii) $J\ y_0=0$ iff $y_0=0$;

(iv) $B(\ J\ y_0, y_0\ )\geq 0\ \ \forall y_0\in F_0$;

(v) $B(\ J\ y_0, y_0\ )=0$ iff $y_0=0$;

(vi) $B(\ x_0,\ J^{-1}x_0\ )=0$ iff $x_0=0$;

(vii) $B(\ x_0,\ y_0\ )=B(\ Jy_0,\ J^{-1}x_0\ ), \forall x_0\in E_0, \forall\ y_0\in F_0$.

Note that if $E\subset F$ and F is a Hilbert space with inner product $\langle x,y\rangle$ we can define $B(x,y)=\langle x,y\rangle$.

Under assumptions (I) and (II),the operator equation (3.2) is equivalent to

$$x=Px+H(I-Q)Nx+JQNx.$$

Theorem 3.2 [7] : Assume that hypotheses (I) and (II) hold.In addition assume that H is compact and N maps bounded sets into bounded sets.Finally,suppose that there exists numbers $R>R_0>0$ such that

(a) the set

$C(R)=\{x_1\in E_1: x_1=\lambda H(I-Q)N(x_0+x_1)$ for some $\lambda\in[0,1]$ and $x_0\in E_0$ with $\|x_0\|\leq R$

is bounded.

(b)$B(x_0, QN(x_0+x_1))\leq 0, x=x_0+x_1, \|x_0\|=R_0, x_1=\lambda H(I-Q)N(x_0+x_1)$ for some $\lambda\in[0,1]$

Then Equation (3.2) has at least one solution.

For the PBVP (1.1)-(1.2), we consider

$$E=\left\{x\in C^1\left[0,2\pi\right]: x(0)=x(2\pi), x^{/}(0)=x^{/}(2\pi)\right\}$$

and $F = L_2[0,2\pi]$.Define $L : D(L) \subset E \longrightarrow F$ by $Lx = x^{//}$, in which $D(L) = \{x \in E : x \in C^2[0,2\pi]\}$ and $N : E \longrightarrow F$ by $Nx = h(t) - f(x)\left|x^{/}\right| - g(x)$.The projections $P : E \longrightarrow E$ and $Q : F \longrightarrow F$ are given by $Px = x(0)$ and $Qx = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt$.Finally,the operator $H : F_1 \longrightarrow E_1$ may be defined by $Hy = x$ if and only if $x^{//} = y$, $x(0) = x(2\pi) = 0$ $x^{/}(0) = x^{/}(2\pi) = 0$.Further,

$$E_0 = \ker L = \{x \in E : x = x(0)\}.$$

$$Range\ L = \left\{y \in F : Qy = \int_0^{2\pi} y(t)dt = 0\right\}.$$

It is easy to see that the hypothesis (I) is satisfied.Now we define $B : E \times F \longmapsto \mathbb{R}$ and $J : F_0 \longmapsto E_0$ respectively by $B(x,y) = \int_0^{2\pi} x(t)\, y(t)dt$ and $Jy_0 = y_0$,so that (II) is satisfied. Clearly H is compact (the inclusion of H^2 into C^1 is compact [see 2]) and N maps bounded sets into bounded sets.We now verify the Conditions (a) and (b) of Theorem3.2.

Let $R > 0$ and $x_1 \in C(R)$,so that $x_1 = \lambda H(I-Q)N(x_0 + x_1)$ for some $\lambda \in [0,1]$ and $x_0 \in E_0$.Hence, $x_1^{//} = \lambda(I-Q)N(x_0 + x_1)$.Multiplying by $x_1^{//}$, we get

$$B(x_1^{//}, x_1^{//}) = \lambda B((I-Q)N(x_0 + x_1), x_1^{//})$$

$$= \lambda B((I-Q)h, x_1^{//}) - \lambda B((I-Q)f(x_0 + x_1)\left|(x_0 + x_1)^{/}\right|, x_1^{//})$$

$$-\lambda B((I-Q)g(x_0 + x_1), x_1^{//}) \tag{3.6}$$

we have

$$B((I-Q)h, x_1^{//}) = B(h, x_1^{//}) - B(Qh, x_1^{//})$$

and

$$B((I-Q)g(x_0 + x_1), x_1^{//}) = B(g(x_0 + x_1), x_1^{//}) - B(Qg(x_0 + x_1), x_1^{//})$$

since $x_1^{//} \in F_1$,then

$$B(Qh, x_1^{//}) = 0,\ B(Q\ g(x_0 + x_1), x_1^{//}) = 0 \text{ and } B(Q\ f(x_0 + x_1)\left|(x_0 + x_1)^{/}\right|, x_1^{//}) = 0$$

and hence (3.6) becomes

$$\begin{aligned} B(x_1^{//}, x_1^{//}) &= \lambda B(h, x_1^{//}) - \lambda B(\ f(x_0 + x_1)\left|(x_0 + x_1)^{/}\right|, x_1^{//}) - \lambda B(\ g(x_0 + x_1), x_1^{//}) \\ &= \lambda B(h, x_1^{//}) - \lambda B(\ f(x_0 + x_1)\left|x_1^{/}\right|, x_1^{//}) - \lambda B(\ g(x_0 + x_1), x_1^{//})\ ,\ (\ x_0 \in E_0 \text{ is a constant}) \end{aligned}$$

and this in turn yields

$$\left\|x_1^{//}\right\|^2 = \lambda \left\langle h, x_1^{//} \right\rangle - \lambda \left\langle f(x_0 + x_1) \left|x_1^{/}\right|, x_1^{//} \right\rangle$$

$-\lambda \left\langle\ g(x_0 + x_1), x_1^{//} \right\rangle$, where $\langle , \rangle$ denotes the usual inner product in L_2 .

Using the Cauchy-Schwartz inequality we get

$$\left\|x_1^{//}\right\|^2 \leq \lambda \left\|h\right\| \left\|x_1^{//}\right\| + \lambda \left\|\ f(x_0 + x_1) \left|x_1^{/}\right|\right\| \left\|x_1^{//}\right\| + \lambda \left\|\ g(x_0 + x_1)\right\| \left\|x_1^{//}\right\|$$

$\lambda \in [0, 1]$, then

$$\left\|x_1^{//}\right\| \leq \left\|h\right\| + \left\|\ f(x_0 + x_1)\right\| \left\|x_1^{/}\right\| + \left\|\ g(x_0 + x_1)\right\| . \tag{3.7}$$

Since $\int_0^{2\pi} x_1^{/}(t)dt = 0$,by Wirtinger's inequality

$$\int_0^{2\pi} (x_1^{/}(t))^2 dt \leq \int_0^{2\pi} (x_1^{//}(t))^2 dt$$

(ie $\left\|x_1^{/}\right\| \leq \left\|x_1^{//}\right\|$) [see 1] and from(3.7) we have

$$(1 - \left\|\ f(x_0 + x_1)\right\|) \left\|x_1^{/}\right\| \leq \left\|h\right\| + \left\|\ g(x_0 + x_1)\right\| = A.$$

If $1 - \left\|\ f(x_0 + x_1)\right\| \neq 0$, then

$$\left\|x_1^{/}\right\| \leq \frac{A}{1 - \left\|\ f(x_0 + x_1)\right\|} = B. \tag{3.8}$$

Using(3.8) in the identity $x_1(t) = x_1(0) + \int_0^t x_1^{/}(s)ds$ (with $x_1(0) = 0$),we get

$$\begin{aligned} |\ x_1(t)| &\leq \int_0^t \left|x_1^{/}(s)\right| ds \leq \int_0^{2\pi} \left|x_1^{/}(s)\right| ds \leq \sqrt{2\pi}(\int_0^{2\pi} \left|x_1^{/}(t)\right|^2 dt)^{\frac{1}{2}} = \sqrt{2\pi} \left\|x_1^{/}\right\| \\ &\leq \sqrt{2\pi} B = C \end{aligned} \tag{3.9}$$

From(3.9),(3.8) and (3.7) we obtain that $x_1, x_1^{/}$ and $x_1^{//}$ are bounded in L_2 and so $C(R)$ is bounded in $H^2(0, 2\pi)$,and in consequence, $C(R)$ is bounded in $C^1[0, 2\pi]$ and in E independently of

$R > 0$.Thus,Condition(a) of Theorem3.2 is satisfied.To verify the Condition (b) of Theorem 3.2 we have

$$\begin{aligned} B(x_0, QN(x_0+x_1)) &= \int_0^{2\pi} x_0 \left[h(t) - f(x_0+x_1) \left| x_1^{/} \right| - g(x_0+x_1) \right] dt \\ &= 2\pi x_0 \, QN(x_0+x_1) \end{aligned}$$

Then Condition (b) of Theorem3.2 is equivalent to the following condition: there exists $R_0 > 0$ such that

$$QN(R_0 + x_1) \leq 0 \leq QN(-R_0 + x_1) \tag{3.10}$$

for every x_1 such that $x_1 = \lambda H(I-Q)N(x_0+x_1)$.

By the above arguments we have proved the following theorem

Theorem 3.3: Suppose that g is increasing,$1 - \| f(x_0+x_1)\| \neq 0$ and condition (3.10) holds.Then the PBVP(1.1)-(1.2) has at least one solution.

Now assume that $\lim_{x \longrightarrow \pm\infty} g(x) = g(\pm\infty)$ exists and

$$g(-\infty) \leq g(x) \leq g(\infty), \text{ for every } x \in \mathbb{R} \tag{3.11}$$

The well-known Landesman-Lazer[6] is

$$g(-\infty) < \omega < g(\infty). \tag{3.12}$$

Corollary 3.4: If g satisfies (3.11), $1 - \| f(x_0+x_1)\| \neq 0$ and (3.12) holds,then the PBVP(1.1)-(1.2) has at least one solution.

Proof: We have from (3.8) and (3.9)

$$\|x_1\|_E = \sup_{t \in [0,2\pi]} | \, x_1(t)| + \left\| x_1^{/} \right\| \leq B + C = B(1+\sqrt{2\pi}) = \delta.$$

Thus, we get $-\delta \leq x_1(t) \leq \delta$, for every $x_1 \in C(R_0), t \in [0, 2\pi]$ and $R_0 > 0$.From (3.12),there exists $T > 0$ such that

$$g(-x) < \omega < g(x) \ for \ x > T. \tag{3.13}$$

Choose $R_0 \geq \delta + T$ so that $R_0 + x_1(t) \geq T$ and $-R_0 + x_1(t) \leq -T$.By integration of (3.13) we get

$$\int_0^{2\pi} [h(t) - g(R_0 + x_1(t))] \, dt \leq 0 \leq \int_0^{2\pi} [h(t) - g(-R_0 + x_1(t))] \, dt,$$

and thus condition (3.10) of Theorem 3.3 is satisfied.

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