Applied Mathematical Sciences, Vol. 2, 2008, no. 22, 1053 - 1062

Viscosity Approximative Methods for Nonexpansive Nonself-Mappings without Boundary Conditions in Banach Spaces

Rabian Wangkeeree and Pramote Markshoe[1]

Department of Mathematics, Faculty of Science
Naresuan University, Phitsanulok 65000, Thailand
rabianw@nu.ac.th (R. Wangkeeree)
pramotem@nu.ac.th (P. Markshoe)

Abstract. Let C be a nonempty closed convex subset of a uniformly smooth Banach space E, $T : C \longrightarrow E$ be a nonexpansive mapping and P be a sunny nonexpansive retraction of E onto C. For $x_0 \in C$, the explicit iterative sequence $\{x_n\}$ is ginven by

$$x_{n+1} = P\left(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)\right) \text{ for } n = 0, 1, 2, \dots,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $[0,1)$ respectively satisfying appropriate conditions, and $f : C \longrightarrow C$ is a fixed contractive mapping. We prove that $\{x_n\}$ converges strongly to a fixed point of T without boundary conditions. The results presented extend and improve the corresponding ones announced by Chen et al. [2], and others.

Mathematics Subject Classification: 47H10, 47H09

Keywords: Nonexpansive nonself-mapping, sunny nonexpansive retraction, uniformly smooth Banach space

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E, and let $T : C \longrightarrow C$ be a nonexpansive mapping (i.e., $\|Tx - Ty \leq \|x - y\|$ for all $x, y \in C$). We use $Fix(T)$ to denote the set of fixed points of T; that is , $Fix(T) = \{x \in C : x = Tx\}$. Recall that a selfmapping $f : C \longrightarrow C$ is a contraction on C if there exists a constant $\beta \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \beta\|x - y\|, \ \forall x, y \in C. \tag{1.1.1}$$

[1]The author is supported in part by Faculty of Science, Naresuan University, Thailand.

Xu [8] defined the following two viscosity iterations for nonexpansive mappings:

$$x_t = tf(x_t) + (1-t)Tx_t \tag{1.1.2}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n \tag{1.1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Xu proved the strong convergence of $\{x_t\}$ defined by (1.1.2) as $t \longrightarrow 0$ and $\{x_n\}$ defined by (1.1.3) in both Hilbert space and uniformly smooth Banach space.

Recently, Song and Chen [4] proved if C is a closed subset of a real reflexive Banach space E which admits a weakly sequentially continuous duality mapping from E to E, and if $T : C \longrightarrow E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition, $F(T) \neq \emptyset$, $f : C \longrightarrow C$ is a fixed contractive mapping, and P is a sunny nonexpansive retraction of E onto C, then the sequences $\{x_t\}$ and $\{x_n\}$ defined by

$$x_t = P(tf(x_t) + (1-t)Tx_t) \tag{1.1.4}$$

and

$$x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n) \tag{1.1.5}$$

strongly converge to a fixed point of T. Very recently, Chen and Zhu [2] established the strong convergence of both $\{x_t\}$ and $\{x_n\}$ defined by (1.1.4) and (1.1.5) respectively, for a nonexpansive nonself-mapping T in a uniformly smooth Banach space.

Let C be a nonempty closed convex subset of a uniformly smooth Banach space E, $T : C \longrightarrow E$ be a nonexpansive nonself-mapping and P be a sunny nonexpansive retraction of E onto C, the purpose of this paper is to use the following iterative process : $x_0 \in C$,

$$x_{n+1} = P\left(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)\right) \text{ for } n = 0, 1, 2, \dots , \tag{1.1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $[0,1)$ respectively, and $f : C \longrightarrow C$ is a fixed contractive mapping, to approximate to the fixed point of nonexpansive mapping T without boundary conditions . Our results extend and improve the corresponding ones announced by Chen et al. [2], and others.

2. Preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\| = \|x\| = \|f\|\}, \forall x \in E$$

where E^* be the dual space of E and and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequence, we will denote the single-valued duality mapping by j, and $x_n \longrightarrow x$ will denote strong convergence of the sequence $\{x_n\}$ to x.

In Banach space E, the following result (*the Subdifferential Inequality*) is well known (Theorem 4.2.1 of [5]): $\forall x, y \in E, \forall j(x+y) \in J(x+y), \forall j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x)\rangle \leq \|x+y\|^2 \leq \|x\|^2 + \langle y, j(x+y)\rangle. \tag{2.2.1}$$

Recall that the norm of E is said to be Gâteaux differentiable if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2.2}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such a Banach space E is called *smooth*. The norm of a Banach space E is also said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit of (2.2.2) is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Frèchet differentiable (and E is said to be uniformly smooth) if the limit in (2.2.2) is attained uniformly for $(x, y) \in U \times U$. A Banach space E is said to be smooth if and only if J is single valued. It is also well known that if E is uniformly smooth, J is uniformly norm-to-norm continuous. These concepts may be found in [5].

If C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \longrightarrow D$ is called a retraction from C to D if $P^2 = P$. It is easily known that a mapping $P : C \longrightarrow D$ is retraction, then $Px = x$, for all $x \in D$. A mapping $P : C \longrightarrow D$ is called sunny if

$$P(Px + t(x + Px)) = Px, \forall x \in C, \tag{2.2.3}$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D. For more detail, see [5]

The following lemma is well known [5].

Lemma 2.1. *Let C be a nonempty convex subset of a smooth Banach space E, $D \subset C$, $J : E \longrightarrow E^*$ the (normalized) duality mapping of E, and $P : C \longrightarrow D$ a retraction. Then the following are equivalent:*

(i) $\langle x - Px, j(y - Px)\rangle \leq 0$ *for all* $x \in C$ *and* $y \in D$

(ii) *P is both sunny and nonexpansive.*

Let C be a nonempty convex subset of a Banach space E, then for $x \in C$, the inward set is given by [6, 7]

$$I_C(x) = \{y \in E : y = x + \lambda(z - x), z \in C, \lambda \geq 0\}. \tag{2.2.4}$$

A mapping $T : C \longrightarrow E$ is said to be satisfying the inward condition if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be satisfying the weakly inward condition if for each $x \in C, Tx \in \overline{I_C(x)}$ where $I_C(x)$ is the closure of $I_C(x)$. Very recently for a nonself-mapping T from C into E, Matsushita and Takahashi [3] studied the following condition:

$$Tx \in S_x^c \tag{2.2.5}$$

for all $x \in C$, where $S_x = \{y \in C : y \neq x, Py = x\}$ and P is a sunny nonexpansive retraction from E onto C. Then they proved the following three lemmas.

Lemma 2.2. [3, Lemma 3.1] *Let C be a closed convex subset of a smooth Banach space E and let T be a mapping form C into E. Suppose that C is a sunny nonexpansive retract of E. If T satisfies the condition (2.2.5), then $F(T) = F(PT)$, where P is a sunny nonexpansive retraction from E onto C.*

Lemma 2.3. [3, Lemma 3.2] *Let C be a closed convex subset of a smooth Banach space E and let T be a mapping form C into E. Suppose that C is a sunny nonexpansive retract of E. If T satisfies the weakly inward condition, then T satisfies the condition (2.2.5).*

Lemma 2.4. [3, Lemma 3.3] *Let C be a closed convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping from C into E. Suppose that C is a sunny nonexpansive retract of E. If $F(T) \neq \emptyset$ then T satisfies the condition (2.2.5).*

The following lemma can be founded in [1].

Lemma 2.5. [1] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\gamma_n\}$ a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\{\beta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\{\alpha_n\}$ a sequence of real numbers with $\limsup_{n\longrightarrow\infty} \alpha_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\alpha_n + \beta_n$$

for all $n \in \mathbb{N}$. Then $\lim_{n\longrightarrow\infty} s_n = 0$.

The following lemma can be founded in [8].

Theorem 2.6. [8] *Let X be a uniformly smooth Banach space, C a closed convex subset of X, $T : C \longrightarrow C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f : C \longrightarrow C$ a contractive mapping. Then as $t \longrightarrow 0$, $\{x_t\}$ defined by*

$$x_t = tf(t) + (1-t)Tx_t \tag{2.2.6}$$

converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (f-I)q, j(q-u)\rangle \leq 0 \text{ for all } u \in F(T). \tag{2.2.7}$$

3. Main Results

Theorem 3.1. *Let X be a uniformly smooth Banach space, C a closed convex subset of X. Suppose that C is a sunny nonexpansive retract of E with P a nonexpansive retraction. Let $T : C \longrightarrow E$ a nonexpansive nonself-mapping with $Fix(T) \neq \emptyset$, and $f : C \longrightarrow C$ be a contractive mapping. Then as $t \longrightarrow 0$, $\{x_t\}$ defined by*

$$x_t = tf(t) + (1-t)PTx_t \tag{3.3.1}$$

converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (f-I)q, j(q-u)\rangle \leq 0 \text{ for all } u \in F(T). \tag{3.3.2}$$

Proof. Applying the Theorem 2.6 with the nonexpansive self-mapping PT, we obtain that $\{x_t\}$ converges strongly as $t \longrightarrow 0$ to a fixed point of PT. Since $F(T) \neq \emptyset$, using Lemma 2.2 and 2.4, we obtain $F(T) = F(PT)$. The proof is complete. □

Theorem 3.2. *Let E be a uniformly smooth Banach space, C is a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let $T : C \longrightarrow E$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$, and $f : C \longrightarrow C$ a fixed contractive mapping with coefficient $\beta \in (0,1)$. The sequence $\{x_n\}$ is defined by (1.1.6), where P is the sunny nonexpansive retraction of E onto C, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$, and satisfying the following conditions:*

(i) $\lim_{n\longrightarrow\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\lim_{n\longrightarrow\infty} \beta_n = 0$;
(iv) $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < +\infty$;
(v) *either* $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty$ *or* $\lim_{n\longrightarrow\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then as $n \longrightarrow \infty$, the sequence $\{x_n\}$ converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the variational inequality (3.3.2).

Proof. First we show that $\{x_n\}$ is bounded. Take $u \in F(T)$, it follows that

$$\begin{aligned}
\|x_{n+1} - u\| &= \|P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)) - Pu\| \\
&\leq \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n) - u\| \\
&\leq \alpha_n\|f(x_n) - u\| + (1-\alpha_n)\|\beta_n(x_n - u) + (1-\beta_n)(Tx_n - u)\| \\
&\leq \alpha_n\beta\|x_n - u\| + \alpha_n\|f(u) - u\| + (1-\alpha_n)\beta_n\|x_n - u\| \\
&\quad +(1-\beta_n)(1-\alpha_n)\|x_n - u\| \\
&= (1-(1-\beta_n)\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\
&\leq \max\{\|x_n - u\|, \frac{1}{1-\beta}\|f(u) - u\|\}.
\end{aligned}$$

By induction, we have

$$\|x_n - u\| \leq \max\{\|x_0 - u\|, \frac{1}{1-\beta}\|f(u) - u\|\}, \forall n \geq 0.$$

Therefore $\{x_n\}$ is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$. Then we get that

$$\begin{aligned}
\|x_{n+1} - PTx_n\| &= \|P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)) - PTx_n\| \\
&\leq \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n) - Tx_n\| \\
&\leq \alpha_n\|f(x_n) - Tx_n\| + (1-\alpha_n)\|\beta_n x_n + (1-\beta_n)Tx_n - Tx_n\| \\
&= \alpha_n\|f(x_n) - Tx_n\| + (1-\alpha_n)\beta_n\|x_n - Tx_n\|
\end{aligned}$$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.3.3}$$

Next we shall show that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3.4}$$

Indeed we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)) \\
&\quad - P(\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})(\beta_{n-1}x_{n-1} + (1-\beta_{n-1})Tx_{n-1}))\| \\
&\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + (1-\alpha_n)\|\beta_n x_n + (1-\beta_n)Tx_n - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})Tx_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|\beta_{n-1}x_{n-1} + (1-\beta_{n-1})Tx_{n-1}\| \\
&\leq \alpha_n\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + (1-\alpha_n)[\beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\|] + |\alpha_n - \alpha_{n-1}|\|\beta_{n-1}x_{n-1} - (1-\beta_{n-1})Tx_{n-1}\| \\
&= \alpha_n\beta\|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|[\|f(x_{n-1})\| \\
&\quad + \|\beta_{n-1}x_{n-1} - (1-\beta_{n-1})Tx_{n-1}\|] + (1-\alpha_n)|\beta_n - \beta_{n-1}|[\|x_{n-1}\| + \|Tx_{n-1}\|] \\
&= (1-(1-\beta)\alpha_n)\|x_n - x_{n-1}\| + K_n.
\end{aligned}$$

where $K_n = |\alpha_n - \alpha_{n-1}|[\|f(x_{n-1})\| + \|\beta_{n-1}x_{n-1} - (1-\beta_{n-1})Tx_{n-1}\|] + (1-\alpha_n)|\beta_n - \beta_{n-1}|[\|x_{n-1}\| + \|Tx_{n-1}\|]$. Since $\{x_n\}$ is bounded, there exists a positive constant K such that

$$K_n \leq K(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|),$$

thus,

$$\|x_{n+1} - x_n\| \leq (1-(1-\beta)\alpha_n)\|x_n - x_{n-1}\| + K(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \tag{3.3.5}$$

Assume that $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty$. By Lemma 2.5 and the conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ we get the required result.

Assume that $\lim_{n\longrightarrow\infty}(\alpha_{n+1}/\alpha_n) = 1$. Then from (3.3.5), we have

$$\|x_{n+1} - x_n\| \leq (1-(1-\beta)\alpha_n)\|x_n - x_{n-1}\| + \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|K + K|\beta_n - \beta_{n-1}|. \tag{3.3.6}$$

By Lemma 2.5 and the conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ we also get the required result. Using (3.3.3) and (3.3.4), we get

$$\|x_n - PTx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - PTx_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.3.7}$$

Let $q = \lim_{t\longrightarrow 0} x_t$, where $\{x_t\}$ is defined in Theorem 3.1, we get that q is the unique solution in $F(T)$ the following variational inequality:

$$\langle (f-I)q, j(q-u)\rangle \leq 0 \text{ for all } u \in F(T). \tag{3.3.8}$$

Next we shall show that

$$\limsup_{n\longrightarrow\infty} \langle f(q) - q, j(x_n - q)\rangle \leq 0. \tag{3.3.9}$$

From (3.3.1) we can write

$$x_t - x_n = t(f(x_t) - x_n) + (1-t)(PTx_t - x_n). \tag{3.3.10}$$

It follows from (3.3.7) that

$$b_n(t) = \|x_n - PTx_n\|(\|x_n - PTx_n\| + 2\|x_n - x_t\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.3.11}$$

Using the inequality (2.2.1), we have

$$\begin{aligned}
\|x_t - x_n\|^2 &\le (1-t)^2\|PTx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\
&\le (1-t)^2\|PTx_t - PTx_n + PTx_n - x_n\|^2 + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle \\
&\quad + 2t\|x_t - x_n\|^2 \\
&\le (1-t)^2\|x_t - x_n\|^2 + (1-t)^2\|x_n - PTx_n\|^2 \\
&\quad + 2(1-t)^2\|PTx_n - x_n\|\|x_t - x_n\| + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle \\
&\quad + 2t\|x_t - x_n\|^2 \\
&\le (1+t)^2\|x_t - x_n\|^2 + b_n(t) + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle. \qquad (3.3.12)
\end{aligned}$$

The last inequality implies

$$\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}b_n(t). \tag{3.3.13}$$

It follows from (3.3.11) that

$$\limsup_{n\longrightarrow\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le M.\frac{t}{2}, \tag{3.3.14}$$

where M is a constant such that $M \ge \|x_t - x_n\|^2$ forall $t \in (0,1)$. By letting $t \longrightarrow 0$ in the last inequality we have

$$\lim_{t\longrightarrow 0}\limsup_{n\longrightarrow\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le 0. \tag{3.3.15}$$

On the other hand, for all $\varepsilon > 0$ there exits a positive δ_1 such that $t \in (0, \delta_1)$,

$$\limsup_{n\longrightarrow\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le \frac{\varepsilon}{2}. \tag{3.3.16}$$

On the other hand, $\{x_t\}$ converges strongly to q, as $t \longrightarrow \infty$, the set $\{x_t - x_n\}$ is bounded, and the duality map J is norm-to-norm uniformly continuous on bounded sets of uniformly smooth space E; from $x_t \longrightarrow q$ as $t \longrightarrow 0$, we get

$$\|f(q) - q - (f(x_t) - x_t)\| \longrightarrow 0 \text{ as } t \longrightarrow 0,$$

and

$$\begin{aligned}
&\|\langle f(q) - q, j(x_n - q)\rangle - \langle f(x_t) - x_t, j(x_n - x_t)\rangle\| \\
= \ &\|\langle f(q) - q, j(x_n - q) - j(x_n - x_t)\rangle + \langle f(q) - q - (f(x_t) - x_t), j(x_n - x_t)\rangle\| \\
\le \ &\|f(q) - q\|\|j(x_n - q) - j(x_n - x_t)\| \\
&+ \|f(q) - q - (f(x_t) - x_t)\|\|j(x_n - x_t)\| \longrightarrow 0 \text{ as } t \longrightarrow 0 \qquad (3.3.17)
\end{aligned}$$

Hence for the above $\varepsilon > 0$, there exists $\delta_2 > 0$ such that for all $t \in (0, \delta_2)$, for all n, we have

$$\|\langle f(q) - q, j(x_n - q)\rangle - \langle f(x_t) - x_t, j(x_n - x_t)\rangle\| \leq \frac{\varepsilon}{2}. \tag{3.3.18}$$

Therefore, we have

$$\langle f(q) - q, j(x_n - q)\rangle\| \leq \langle f(x_t) - x_t, j(x_n - x_t)\rangle + \frac{\varepsilon}{2}. \tag{3.3.19}$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, for all $t \in (0, \delta)$, we have

$$\limsup_{n\longrightarrow\infty}\langle f(q) - q, j(x_n - q)\rangle \leq \limsup_{n\longrightarrow\infty}(\langle f(x_t) - x_t, j(x_n - x_t)\rangle + \frac{\varepsilon}{2}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.3.20}$$

Since ε is arbitrary, we get the required inequality (3.3.9). Finally, we shall show that $x_n \longrightarrow q$ as $n \longrightarrow \infty$. We note that

$$x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n)q) = (x_{n+1} - q) - \alpha_n(f(x_n - q)).$$

Using the inequality (2.2.1), we have,

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n)q) + \alpha_n(f(x_n - q))\|^2 \\
&\leq \|x_{n+1} - P(\alpha_n f(x_n) + (1 - \alpha_n)q)\|^2 + 2\alpha_n\langle f(x_n) - q, j(x_{n+1} - q)\rangle\| \\
&\leq \|x_{n+1} - P(\alpha_n f(x_n) + (1 - \alpha_n)q)\|^2 + 2\alpha_n\langle f(x_n) - q, j(x_{n+1} - q)\rangle\| \\
&\leq \|\alpha_n f(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 + \beta_n)Tx_n) - (\alpha_n f(x_n) + (1 - \alpha_n)q)\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(q), j(x_{n+1} - q)\rangle\| + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle\| \\
&\leq (1 - \alpha_n)^2[\beta_n\|x_n - q\| + (1 + \beta_n)\|Tx_n - q\|]^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(q), j(x_{n+1} - q)\rangle\| + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle\| \\
&\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\|f(x_n) - f(q)\|\|x_{n+1} - q\| \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle\| \\
&\leq (1 - \alpha_n)^2\|x_n - q\|^2 + \alpha_n(\|f(x_n) - f(q)\|^2 + \|x_{n+1} - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle.
\end{aligned}$$

Therefore we have

$$(1-\alpha_n)\|x_{n+1}-q\|^2 \leq (1-2\alpha_n+\alpha_n^2)\|x_n-q\|^2+\alpha_n\beta\|x_n-q\|^2+2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle.$$

Thus,

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq (1 - \frac{1 - \beta^2}{1 - \alpha_n}\alpha_n)\|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n}\|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n}\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq (1 - \gamma_n)\|x_n - q\|^2 + \lambda\gamma_n\alpha_n + \frac{2}{1 - \beta_n^2}\gamma_n\langle f(q) - q, j(x_{n+1} - q)\rangle,
\end{aligned}$$

where $\gamma_n = \frac{1-\beta^2}{1-\alpha_n}\alpha_n$ and λ is a constant such that $\lambda > \frac{1}{1-\beta_n}\|x_n - q\|^2$. Hence

$$\|x_{n+1} - q\|^2 \leq (1 - \gamma_n)\|x_n - q\|^2$$

$$+\gamma_n(\lambda\alpha_n + \frac{2}{1-\beta_n^2}\gamma_n\langle f(q)-q, j(x_{n+1}-q)\rangle). \quad (3.3.21)$$

It is easily seen that $\gamma_n \longrightarrow 0, \sum_{n=1}^{\infty}\gamma_n = \infty$, and noting that

$$\lim_{n\longrightarrow\infty}(\lambda\alpha_n + \frac{2}{1-\beta_n^2}\gamma_n\langle f(q)-q, j(x_{n+1}-q)\rangle) \leq 0.$$

Applying Lemma 2.5 onto (3.3.21), we have $\{x_n\}$ converges strongly to q. The proof is complete. □

If in Theorem 3.2 , $\beta_n = 0$ for all $n \geq 0$, then the iteration (1.1.6) reduces to the iteration (1.1.5). Note that, the weakly inward conditions on the mapping T can be dropped. In fact, the following Corollary can be obtained from Theorem 3.2 immediately.

Corollary 3.3. [2, Theorem 3.4] *Let E be a uniformly smooth Banach space, C is a nonempty closed convex subset of E, let $T : C \longrightarrow E$ be a nonexpansive nonself-mapping satisfying the weakly inward conditions, and $F(T) \neq \emptyset$. Let $f : C \longrightarrow C$ a fixed contractive mapping. The sequence $\{x_n\}$ is defined by (1.1.5), where P is the sunny nonexpansive retraction of E onto C, and $\{\alpha_n\} \subset (0,1)$, and satisfying the following conditions:*

(i) $\lim_{n\longrightarrow\infty}\alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iii) *either* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < +\infty$ *or* $\lim_{n\longrightarrow\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then as $n \longrightarrow \infty$, the sequence $\{x_n\}$ converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (f-I)q, j(q-u)\rangle \leq 0 \text{ for all } u \in F(T).$$

If in Theorem 3.2 , $T : C \longrightarrow C$ is the nonexpansive mapping and $\beta_n = 0$ for all $n \geq 0$, then the iteration (1.1.6) reduces to the iteration (1.1.3). In fact, the following Corollary can be obtained from Theorem 3.2 immediately.

Corollary 3.4. [8, Theorem 4.2] *Let E be a uniformly smooth Banach space, C is a nonempty closed convex subset of E, let $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f : C \longrightarrow C$ a fixed contractive mapping. The sequence $\{x_n\}$ is defined by (1.1.3) and $\{\alpha_n\} \subset (0,1)$ satisfying the following conditions:*

(i) $\lim_{n\longrightarrow\infty}\alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iii) *either* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < +\infty$ *or* $\lim_{n\longrightarrow\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then as $n \longrightarrow \infty$, the sequence $\{x_n\}$ converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (f-I)q, j(q-u)\rangle \leq 0 \text{ for all } u \in F(T).$$

References

[1] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda., Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Analysis (2006), doi:10.1016/j.na.2006.08.032

[2] R. Chen and Z. Zhu, Strong convergence of approximation fixed point for nonexpansive nonself-mapping, International Journal of Mathematics and Mathematical Sciences (20006), Article ID 16470, P. 1 -12, DOI 10.1155/IJMMS/2006/16470.

[3] S. Matsushita and W. Takahashi, Strong convergence theorem for nonexpansive nonself-mappings without boundary conditions, Nonlinear Analysis (2006), doi:10.1016/j.na.2006.11.007.

[4] Y. Song and R. Chen, Viscosity approximation methods for nonexpansive nonself-mappings, Journal of Mathematical Analysis and Applications 321 (2006), no. 1, 316 - 326.

[5] W. Takahashi, Nonlinear Functional Analysis? Fixed Point Theory and its Applications, Yokohama Publishers Inc., Yokohama, 2000 (in Japanese).

[6] W. Takahashi and G. E. Kim, Strong convergence of approximants to fixed points of non-expansive nonself-mappings in Banach space, Nonlinear Anal., 32 (1998) 447-454.

[7] H.-K. Xu, Approximating curves of nonexpansive nonself-mappings in Banach spaces, Comptes Rendus de l'Academie des Sciences. Serie I. Mathematique 325 (1997), no. 2, 151 - 156.

[8] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings,* J. Math. Anal. Appl. **298** (2004) 279-291.

Received: September 29, 2007