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Ranking Decision Making Units in Fuzzy-DEA Using Entropy

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Abstract

Data Envelopment Analysis (DEA) can be regarded as a useful management tool to the assessment evaluation of decision making units (DMUs) using multiple inputs to produce multiple outputs. In some cases, to evaluate the efficiency having imprecise inputs and outputs such as fuzzy or interval data the efficiency of DMUs won't be exact as well. Most researches have been conducted were based on getting define efficiency and ranking so far. In this paper, the fuzzy efficiency scores of decision making units are counted and the entropy of which is determined and at the same time, they will be ranked from a new view point. To do this, maximum entropy as a special class weighting function is used, and then the fuzzy efficiency of DMUs considering the optimistic level will be computed. At the end, having a numerical example, the concept of method analyzed.

Mathematics Subject Classification: 90C70

Keywords: DEA; Maximum entropy; Ranking; Weighting function

1 Introduction

Data envelopment analysis (DEA), as a very useful management and decision tool, has found surprising development in theory and methodology and extensive applications in the range of the whole world since it was first developed by Charnes et al. Traditional DEA models require crisp input/output data. However, in real-world problems inputs and outputs are often imprecise. Most of the previous studies that deal with inexact and imprecise inputs and outputs in DEA models have simply used simulation techniques like the one in Banker et al.[1]. Cooper et al.[3] studied how to deal with imprecise data .The final efficiency score for each DMU was derived as a deterministic numerical value less than or equal to unity. In recent years, fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models. Sengupta [12] was the first to introduce a fuzzy DEA model. The DEA models with fuzzy data ("fuzzy DEA" models) can more realistically represent real-world problems than the conventional DEA models. Fuzzy set theory also allows linguistic data to be used directly within the DEA models. Fuzzy DEA models take the form of fuzzy linear programming models. A typical approach to fuzzy linear programming requires a method to rank fuzzy sets and different fuzzy ranking methods may lead to different results [11]. The problem of ranking fuzzy sets has been addressed by many researchers. Good references on fuzzy ranking methods and properties thereof are in Dubois and Prade [4]. As pointed by Wang and Kerre[14].

Our aim in this paper is to explore the application entropy fuzzy in fuzzy DEA model, and the ranking of decision making units (DMUs).

Entropy of fuzzy set describes the fuzziness degree of fuzzy set. Many scholars have studied it from different points of view. For example, in 1972, De Luca and Termini[5] introduced some axioms to describe the fuzziness degree of fuzzy set. Kaufmann [10] proposed a method to measure the fuzziness degree of fuzzy set by a metric distance between its membership function and the membership function of its nearest crisp set. Another way given by Yager [15] was to view the fuzziness degree of fuzzy set in terms of a lack of distinction between the fuzzy set and its complement. Some authors have investigated interval valued fuzzy set and it's some relevant topics, for example, in 2004, Grzegorzewski [8] studied distance between interval valued fuzzy sets based on the Hausdroff metric, Burillo and Bustince[2] and Szmidt et al.[13] researched entropy of interval valued fuzzy set from different point of views, respectively. Liu[11] expanded centroid method of fuzzy number to a generic form with weighting function.

The rest of this paper is organized as follows: In Sections 2 and 3 we review fuzzy DEA model and entropy fuzzy. In Section 4 we investigated the application entropy in DEA. An illustrative example is presented in Section 5.

2 Fuzzy DEA model

Fuzzy DEA models take the form of fuzzy linear programming models. Consider n DMUs; each consumes varying amounts of m different fuzzy inputs to produce s different fuzzy outputs. In the model formulation, $\widetilde{x_{io}}\ \ i = 1, \cdots, m$ and $\widetilde{y_{ro}}\ \ r = 1, \cdots, s$ denote, respectively, the input and output values forDMU_o, the DMU under consideration. The programming statement for the (input oriented) fuzzy CCR model is:

$$
\begin{aligned}
\max \quad & E_o=\sum_{r=1}^{s} u_r\widetilde{y_{ro}} \\
s.t. \quad & \sum_{i=1}^{m} v_i\widetilde{x_{io}}=\widetilde{1} \\
& \sum_{r=1}^{s} u_r\widetilde{y_{rj}}-\sum_{i=1}^{m} v_i\widetilde{x_{ij}}\leq 0 \qquad j=1,\cdots,n \\
& u_r,v_i\geq 0 \quad r=1,\cdots,s \quad ,i=1,\cdots,m
\end{aligned} \tag{1}
$$

The α-cuts, also known as the α-level sets, of $\widetilde{x_{ij}}$ and $\widetilde{y_{rj}}$ are defined as $(\widetilde{x_{ij}})_\alpha=\{x\in X \mid \mu_{x_{ij}(x)}\geq\alpha\}=[x_{ij}^l,x_{ij}^u]$
and $(\widetilde{y_{rj}})_\alpha=\{x\in X \mid \mu_{y_{rj}(x)}\geq\alpha\}=[y_{rj}^l,y_{rj}^u]$
Applying the $\alpha-level$ of fuzzy DEA, the following model would be achieved:

$$
\begin{aligned}
\max \quad & E_o=\textstyle\sum_{r=1}^{s} u_r[y_{ro}^l,y_{ro}^u] \\
s.t. \quad & \textstyle\sum_{i=1}^{m} v_i[x_{io}^l,x_{io}^u]=\widetilde{1} \\
& \textstyle\sum_{r=1}^{s} u_r[y_{rj}^l,y_{rj}^u]-\sum_{i=1}^{m} v_i[x_{ij}^l,x_{ij}^u]\leq 0 \qquad j=1,\cdots,n \\
& u_r,v_i\geq 0 \quad r=1,\cdots,s \ , \quad i=1,\cdots,m
\end{aligned} \tag{2}
$$

Now interval DEA model is developed for measuring the lower and upper bounds of the best relative efficiency of each DMU with interval input and output data.

$$
\begin{aligned}
\max \quad & (E_o)_\alpha^u=\textstyle\sum_{r=1}^{s} u_r(y_{ro})_\alpha^u \\
s.t. \quad & \textstyle\sum_{i=1}^{m} v_i(x_{io})_\alpha^l=\widetilde{1} \\
& \textstyle\sum_{r=1}^{s} u_r(y_{ro})_\alpha^u-\sum_{i=1}^{m} v_i(x_{io})_\alpha^l\leq 0 \\
& \textstyle\sum_{r=1}^{s} u_r(y_{rj})_\alpha^l-\sum_{i=1}^{m} v_i(x_{ij})_\alpha^u\leq 0 \qquad j=1,\cdots,n \quad j\neq o \\
& u_r,v_i\geq 0 \quad r=1,\cdots,s \ , \quad i=1,\cdots,m
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\max \quad & (E_o)_\alpha^l=\textstyle\sum_{r=1}^{s} u_r(y_{ro})_\alpha^l \\
s.t. \quad & \textstyle\sum_{i=1}^{m} v_i(x_{io})_\alpha^u=\widetilde{1} \\
& \textstyle\sum_{r=1}^{s} u_r(y_{ro})_\alpha^l-\sum_{i=1}^{m} v_i(x_{io})_\alpha^u\leq 0 \\
& \textstyle\sum_{r=1}^{s} u_r(y_{rj})_\alpha^u-\sum_{i=1}^{m} v_i(x_{ij})_\alpha^l\leq 0 \qquad j=1,\cdots,n \quad j\neq o \\
& u_r,v_i\geq 0 \quad r=1,\cdots,s \ , \quad i=1,\cdots,m
\end{aligned} \tag{4}
$$

Theorem 2.1 *for ever* α *,* $E_\alpha^l\leq E_\alpha^u$

Theorem 2.2 *If* $\alpha_1\leq\alpha_2$ *then* $[E_{\alpha_2}^l,E_{\alpha_2}^u]\subseteq[E_{\alpha_1}^l,E_{\alpha_1}^u]$

3 Entropy fuzzy

In physics, the word entropy has important physical implications as the amount of "disorder" of a system; In mathematics, a more abstract definition is used. Entropy is as a measure of probabilistic uncertainty. Concept of entropy has

penetrated a wide range of disciplines, such as statistical mechanics, business, pattern recognition, transportation, information theory, queuing theory, linear and nonlinear programming and so on. To define entropy, Shannon (1948) proposed some axioms. (1)Expansibility (2)Symmetry (3)Continuity (4)Maximum (5)Additivity (6)Monotonicity (7)Branching (8)Normalization. The Shannon entropy of a variable A (discrete set) is defined as $H(A) = -\sum p(x).\ln(p(x))$
where p(x)denotes the probability distribution in the universal set X for all $x \in X$, and the entropy of a continuous probability distribution with the probability density function p(x) as $H(A) = -\int p(x).\ln(p(x))dx$
The first fuzzy entropy formula without reference to probabilities was proposed in 1972 in the work of De Luca and Termini[5], who defined the entropy using Shannon's functional form.
$H(A) = -\sum \mu_A(x)\ln(\mu_A(x)) - (1-\mu_A(x))(\ln(1-\mu_A(x)))$(5)

Definition 3.1 *A real function $H : F \to R^+$ is called entropy on F (fuzzy set), if H has the following properties:*
(P1) H(A)=0 if A is a crisp set (P2) H(A) is a unique maximum if $\mu_A(x) = \frac{1}{2}$ (P3) If A^* is a sharpened version of A, then $H(A^*) \leq H(A)$; fuzzy set A^* a sharpened version of A, if $\mu_{A^*}(x) \geq \mu_A(x)$ when $\mu_A(x) \geq \frac{1}{2}$ and$\mu_{A^*}(x) \leq \mu_A(x)$ when $\mu_A(x) \leq \frac{1}{2}$. (P4) $H(A^c) = H(A)$, where A^c is the standard complement of A, i.e. $\mu_{A^c}(x) = 1 - \mu_A(x)$.

The measure of fuzziness $H(A)$ can be regarded as "entropy" of a fuzzy set A. At a fixed element x , $H(\mu_A(x)) = h(\mu_A(x))$where the entropy function $h : [0,1] \to [0,1]$ is monotonically increasing in [0,0.5] and monotonically decreasing in [0.5,1], moreover h(u) = 0 , as u = 0 and 1; and h(u) = 1, as u= 0.5 . Some well-known entropy functions are shown in the following:

$$h(u) = \begin{cases} 2u & u \in [0, \frac{1}{2}] \\ 2(1-u) & u \in [\frac{1}{2}, 1] \end{cases} \quad (6)$$

$h(u) = 4u(1-u)$ (7)
and $h(u) = -u.\ln(u) - (1-u).\ln(1-u)$ (8)
where last is Shannon's function. To determine a global entropy measure of the fuzzy set A independent of x, if A is a discrete fuzzy set, entropy can be defined as $H(A) = \sum_{x\in(X)} h(\mu_A(x))$
and for a continuous fuzzy set A, we can integrate over the universal set X as follows: $H(A) = \int_{x\in(X)} h(\mu_A(x))dx$ (9)
It is known that the larger H(A) is, the more is the fuzziness of the fuzzy set A.

3.1 Entropy of interval valued fuzzy set

In 2001, Szmidt et al.[13] extended De Luca and Termini [5] axioms for fuzzy set to introduce entropy of intuitionistic fuzzy set. Based on this view point of Szmidt et al.[13], Zeng and Shi[16] introduced the concept of entropy of interval valued fuzzy set which is different from Bustince and Burillo [2].

Definition 3.2 *A real function $E : IVFS \rightarrow [0,1]$ is called entropy on $IVFSs$(interval valued fuzzy sets), if E satisfies the following properties:*
(P1) H(A) = 0 iff A is a crisp set
(P2) H(A)=1 iff $\mu_{A^-}(x) + \mu_{A^+}(x)) = 1$
$(P3)H(A) \leq H(B)$ if A is less fuzzy than B, i.e., $\mu_{A^-}(x) \leq \mu_{B^-}(x)$ and $\mu_{A^+}(x) \leq \mu_{B^+}(x)$ for $\mu_{B^-}(x)+\mu_{B^+}(x) \leq 1$ or $\mu_{A^-}(x) \geq \mu_{B^-}(x)$and $\mu_{A^+}(x) \geq \mu_{B^+}(x))$ for $\mu_{B^-}(x) + \mu_{B^-}(x) \geq 1$
$(P4)H(A) = H(A^c)$.

Then we can give the following formulas to calculate entropy of interval valued fuzzy set A:

$$H_1(A) = 1-\frac{1}{n}\sum_{i=1}^{n} \mid \mu_{A^-}(x_i)+\mu_{A^+}(x_i)-1 \mid \quad (10)$$

$$H_2(A) = 1-\sqrt{\sum_{i=1}^{n}(\mu_{A^-}(x_i) + \mu_{A^+}(x_i) - 1)^2} \quad (11)$$

$$H_3(A) = 1-\frac{1}{b-a}\int_a^b \mid \mu_{A^-}(x_i)+\mu_{A^+}(x_i)-1 \mid dx \quad (12)$$

$$H_4(A) = \frac{\int_a^b[\mu_{A^-}(x)\wedge(1-\mu_{A^+}(x))]dx}{\int_a^b[\mu_{A^-}(x)\vee(1-\mu_{A^+}(x))]dx} \quad (13)$$

where the integral in $H3$ and $H4$ is Lebesgue integral.

4 Entropy in DEA

In most of the existing methods for possibilistic linear programming, where the α-cut is used, the solution is obtained by comparing the intervals in left and right hand side of the constraints. Different methodologies have been suggested for the comparison of the intervals. In some of these methods simply the end points of the interval are considered for justification that makes the model very simple and hence a lot of information might have been lost. In the others the complexity of the algorithm may cause computational inefficiency DEA assigns an efficiency score less than one to inefficient DMUs and equal to one to efficient DMUs. So, for inefficient DMUs a ranking is given but for efficient ones no ranking can be given. Some methods for ranking efficient DMUs with crisp data are developed. In this paper, by considering fuzzy DMUs, an alternative ranking method based on entropy of efficiency of DMUs as weighting function is proposed.

Definition 4.1 *The weighting function expectation for fuzzy number A with*

supp(A) $= [a,b]$ *can be defined* $V_f(A) = \frac{\int_a^b x\mu_A(x)f(x)dx}{\int_a^b \mu_A(x)f(x)dx}$ (14)
where f(x) is called weighting $f(x) \geq 0$ *with, and* $\int_a^b f(x)dx \neq 0$.

Definition 4.2 *For weighting function of fuzzy number in [a,b], f(x), the optimistic degree is* $\beta_f = \frac{\int_a^b (x-a)f(x)dx}{(b-a)\int_a^b f(x)dx}$ (15)
As $f(x) \geq 0$, $0 \leq \frac{x-a}{b-a} \leq 1$ $x \in [a,b]$ *so* $0 \leq \beta_f \leq 1$.
Some properties of the optimistic measure and the weighting function are as the following:

Theorem 4.3 *When A is a interval number as [a,b],* $V_f(A)$ *becomes the linear combination of a and b with* β_f. $(V_f(A) = a + (b-a)\beta_f)$

Theorem 4.4 *For weighting function f(x), and any fuzzy number A with* *supp(A)* $= [a,b]$, *if* $\beta_f \to 0 \Rightarrow v_f(A) \to a$ *if* $\beta_f \to 1 \Rightarrow v_f(A) \to b$

As mentioned before, the weighting function can be seen as the decision function representing the decision maker's attitude. In this section, we will propose a special class of weighting functions under given optimistic level with maximum entropy principle. The entropy of weighting function f(x) can be defined as $H_f = \int_a^b 4f(x)(1-f(x))dx$ (16)
with $\int_a^b f(x)dx = 1$.
The maximum entropy weighting function problem with given optimistic level is

$$\max \quad H_f = \int_a^b 4f(x)(1-f(x))dx$$

$$s.t \quad \int_a^b \frac{x-a}{b-a} f(x)dx = \beta \quad 0 < \beta < 1$$

$$\int_a^b f(x)dx = 1 \tag{17}$$

This is a variational optimization problem; the Lagrangian is
$\Lambda(f(x), f'(x), x, \lambda_1, \lambda_2) = 4f(x)(1-f(x)) + \lambda_1 \frac{x-a}{b-a} f(x) + \lambda_2 f(x)$
$4 - 8f(x) + \lambda_1 \frac{x-a}{b-a} f(x) + \lambda_2 f(x) = 0$
Considering the constraints of (17), we can get that
$f(x) = \frac{12\beta-6}{(b-a)^2}(x - \frac{a+b}{2}) + \frac{1}{b-a}$ (18)
if $\beta = \frac{1}{2} \Rightarrow f(x) = \frac{1}{b-a} \Rightarrow V_f(A) = E(x)$
This means the relative position of the preference expectation value should remain when the membership function is translated, i.e., for fuzzy number A with membership function $\mu_A(X)$ and *supp*$(A) = [a,b]$, let B with $\mu_B(x) = \mu_A(x-c)$ (c is a constant), then for given valuation optimistic level β, $V_\beta(B) = V_\beta(A) + c$, where $V_\beta(A)$, $V_\beta(B)$ are the maximum entropy weighting function

expectation of A, B with $\beta_f = \beta$, respectively.
Let $D : \mu_D(x) = \mu_A(\frac{x}{c}) \quad (c \in R - \{0\})$, then

$$V_\beta(D) = \begin{cases} cV_\beta(A) & c > 0 \\ cV_{1-\beta}(A) & c < 0 \end{cases}$$

5 Numerical example

An example with two fuzzy inputs and two fuzzy outputs illustrated in Table 1 is considered. Interval efficiencies obtained from the models (3) and (4) for different α values are listed in Table 2.

Table 1 - DMUs with two fuzzy inputs and two fuzzy outputs

DMU	A	B	C	D	E
X_1	(4,3.5,4.5)	(2.9,2.9,2.9)	(4.9,4.4,5.4)	(4.1,3.4,4.8)	(6.5,5.9,7.1)
X_2	(2.1,1.9,2.3)	(1.5,1.4,1.6)	(2.6,2.2,3.0)	(2.3,2.2,2.4)	(4.2,3.6,4.6)
Y_1	(2.6,2.4,2.8)	(2.2,2.2,2.2)	(3.2,2.7,3.7)	(2.9,2.5,2.3)	(5.1,4.4,5.8)
Y_2	(4.1,3.8,4.4)	(3.5,3.3,3.7)	(5.1,4.3,5.9)	(5.7,5.5,5.9)	(7.4,6.5,8.3)

Table 2 -Efficiency of DMUs

DMU	A	B	C	D	E
$\alpha = 0.0$	[0.654,1]	[0.836,1]	[0.571,1]	[0.855,1]	[0.638,1]
$\alpha = 0.25$	[0.702,1]	[0.908,1]	[0.642,1]	[0.943,1]	[0.735,1]
$\alpha = 0.50$	[0.758,0.963]	[0.99,1]	[0.716,1]	[1,1]	[0.845,1]
$\alpha = 0.75$	[0.807,0.904]	[1,1]	[0.791,0.932]	[1,1]	[0.969,1]
$\alpha = 1.00$	[0.855,0.855]	[1,1]	[0.861,0.861]	[1,1]	[1,1]

To evaluate the entropy of illustrated DMUs, we make membership functions of fuzzy efficiency of DMUs, $\mu(x)$, and then the formula (7) for $h(\mu(x))$ is used. The results computed entropy as (9), listed in table 3.

Table 3- Entropy of DMUs

DMU	A	B	C	D	E
Entropy	0.25217	0.1143	0.28225	0.08892	0.2678

Entropy figures out the uncertainty and fuzziness of score efficiency of DMUs, therefore, the sums concluded of entropy will be more for DMUs having higher or lower efficiency, thus value entropy isn't able to ranking DMUs, But when entropy use to making weighting function results suitable ranking for DMUs. The preference expectations of fuzzy efficiency of DMUs with use weighting function (18) and different valuation of optimistic levels are listed in Table 4.

Table 4 - Fuzzy efficiency of DMUs with different valuation of β

	A	B	C	D	E
$\beta = 0.1$	0.6966	0.93512	0.70114	0.94634	0.7759
$\beta = 0.5$	0.84449	0.9456	0.83035	0.95394	0.87783
$\beta = 0.7$	0.8787	0.9497	0.8642	0.95695	0.90108
$\beta = 0.9$	0.90266	0.95325	0.88881	0.95958	0.91720

Ranking of DMUs is as follow:

Table 5 - The ranking order for the five DMUs under different optimistic measure to β

DMU	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
A	5	4	4	4
B	2	2	2	2
C	4	5	5	5
D	1	1	1	1
E	3	3	3	3

6 Main Results

Since the efficiency being fuzzy in data envelopment analysis models with fuzzy data, it will be difficult to rank the efficiencies. In this paper, an effective method to rank efficient DMUs and inefficient DMUs is suggested. Compare the other methods, it is more stable. This method is an extension of definite class of weight function based on the principle of maximum entropy. Defining a parameter weight function regarding the optimistic view, the efficiency is measured and contrasted. One of the most significant advantages of this method is the compatibility and stability of which in ranking. Regarding the β index, the manager's opinion to measure efficiency has been considered as well. Also this approach has variety application in engineering industrial.

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