A Fuzzy Set Based Framework

for Concept of Affinity

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Abstract. The traditional set theory is constructed on the steady state, the bivalent logic and the fixed set boundary. However, the development of fuzzy set or rough set inspires us that an element, which membership degree to a specified set could be multivalent (fuzzy set) and the set boundary could also be undetermined (rough set). No mattering from the scope of traditional set, fuzzy set or rough set, the element is belonged to a specified set or not is assumed as a steady (time-independent) behavior. Thus, we may regard these aforementioned set theories as the time-independent set theories. In this study, we explore the possibility of proposing a time-dependent set theory, which means the relation between two objects we consider it should be time-dependent: we name it the affinity set theory. We present a simple framework for the concept of affinity based on fuzzy set theory. The notion of affinity set is introduced. A new forecasting method based on game theory and affinity set is also presented.

Keywords: Affinity, fuzzy set, rough set, relation

1. Introduction

A traditional set is any well (precisely) defined collection of "objects." The elements of a set are the objects in that set [4]. Usually we denote sets with upper-case letters, elements with lower-case letters. The following notation is used to show set membership: if we confirm that that x is a member of the set A, then we express this relation by $x \in A$; on the contrary, we use $x \notin A$ to show that x is not a member of the set A. Set theory is the foundation of mathematics: all mathematical concepts can be characterized in terms of the primitive notions of set and membership.

Many aspects of social behavior are based on affinity, especially, the formation and evolution of groups or networks [3, 6]. Affinity has two meanings. The first meaning is that it is natural liking for or attraction to a person, thing, idea, etc., in this paper we call it d*irect affinity*. The second is that affinity is defined as a close relationship between people or things that have similar qualities, structures, properties or features. In this paper we call it *indirect affinity*. When dealing with affinity two difficulties arise. First, affinity is, by definition, a vague and imprecise concept. The second is that affinity often, if not always, varies with time. Thus, a rational theory of affinity should take into account the vagueness (fuzziness) of this concept and its time-dependence. As far as we know, in literature, there is no theory dealing with affinity as vague and time-dependent concept [6]. In this paper we propose a theoretical framework for the concept of affinity based on fuzzy sets and fuzzy relations [2, 5]. Indeed, fuzzy set theory is one of the best tools for representing vague and imprecise concepts like affinity.

This paper is organized as follows: in Section2, we will review the basic concepts of fuzzy set and rough set. In Section 3, we propose our formal definitions of affinity.

In Section 4, a simple forecasting model based on game theory and affinity set theory is proposed. Finally, conclusions and recommendations are in Section 5.

2. Preview of Related Literatures

Here, we will briefly review two popular sets in the management/information science: they are the fuzzy set [2, 5] and the rough set [7].
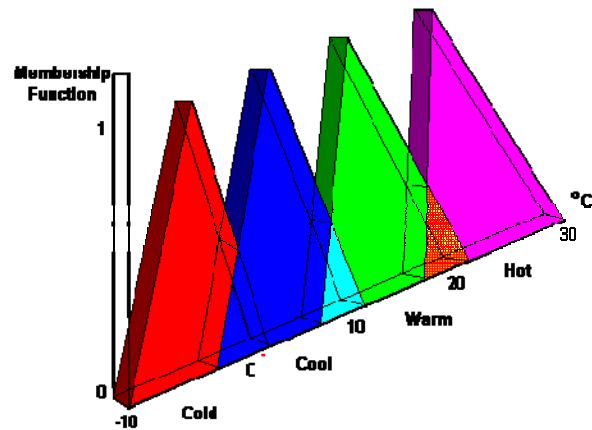
2.1 Fuzzy Set

Bivalent Set Theory (the traditional set theory) can be somewhat limiting if we wish to describe a "humanistic" problem mathematically. For example, Fig 1 below illustrates bivalent sets to characterize the temperature of a room. The most obvious limiting feature of bivalent sets that can be seen clearly from the diagram is that they are mutually exclusive - it is not possible to have membership of more than one set (opinion would widely vary as to whether 50 degrees Fahrenheit is 'cold' or 'cool' hence the expert knowledge we need to define our system is mathematically at odds with the humanistic world). Clearly, it is not accurate to define a transition from a quantity such as 'warm' to 'hot' by the application of one degree Fahrenheit of heat. In the real world a smooth (unnoticeable) drift from warm to hot would occur.
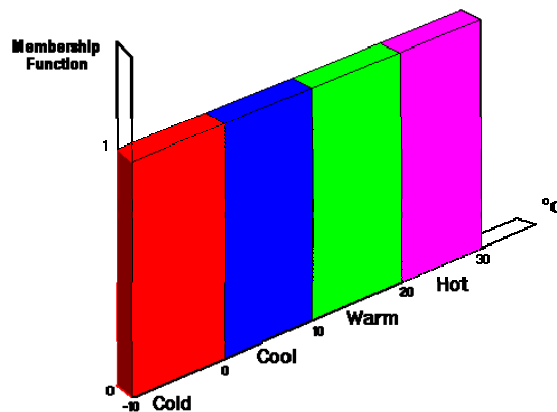
Fig. 1 : Bivalent Sets to Characterize the Temp. of a room.

Source [5]

This natural phenomenon can be described more accurately by Fuzzy Set Theory. Fig.2 below shows how fuzzy sets quantifying the same information can describe this natural drift.

Fig. 2 - Fuzzy Sets to characterize the Temp. of a room.

Source [5]

Def. 2.1 Universe of Discourse

The Universe of Discourse is the range of all possible values for an input to a fuzzy system.

Def. 2.2 Fuzzy Set

A Fuzzy Set is any set that allows its members to have different grades of membership (membership function) in the interval [0,1].

Def. 2.3 Support

The Support of a fuzzy set F is the crisp set of all points in the Universe of Discourse U such that the membership function of F is non-zero.

Def. 2.4 Crossover point

The Crossover point of a fuzzy set is the element in U at which its membership function is 0.5.

Def. 2.5 Fuzzy Singleton

A Fuzzy singleton is a fuzzy set whose support is a single point in U with a membership function of one.

Fuzzy Set Operations.

Op. 2.1 Union

The membership function of the Union of two fuzzy sets A and B with membership functions μ_A and μ_B respectively is defined as the maximum of the two individual membership functions $\mu_{A\cup B} = \max(\mu_A, \mu_B)$.

Op. 2.2 Complement

The membership function of the Complement of a Fuzzy set A with membership function $\mu_{\overline{A}}$ is defined as $\mu_{\overline{A}} = 1 - \mu_A$.

In addition, since fuzzy set is also a set theory, it follows the De Morgan's law, associativity and commutativity in the traditional set theory. With these basic definitions and logic operations, many scholars had developed various fuzzy models in the field of fuzzy set [2, 5].

2.2 Rough Set

Rough set theory was proposed by Zdzislaw Pawlak in 1982 [7]. Since then we have witnessed systematic, world-wide growth of interest in rough set theory and its applications. The theory of rough sets deals with the classificatory analysis of data tables. We define the necessary notations as follows: IS is a pair (U, A), U is a non-empty finite set of objects, and A is a non-empty finite set of attributes such that for every $a: U \to V_a$, here V_a is called the value set of a. This is a classification process that dividing the U into many As according to the attributes of A. Suppose we are given a finite non-empty set U of objects, called universe. Each object of U is characterized by a description, for example a set of attribute values. In a standard rough set model, the knowledge is usually and formally represented by an

equivalence relation IND (called indiscernibility relation) defined on a certain universe of objects U. The pair (U, IND) is called an approximation space. The approximation space provides an approximate characterization of any subset X of U.

Def. 2.5 Indiscernibility

R is a binary relation, which is reflexive ($x\mathrm{R}x$ for any object x), symmetric (if $x\mathrm{R}y$ then $y\mathrm{R}x$), and transitive (if $x\mathrm{R}y$ and $y\mathrm{R}z$ then $x\mathrm{R}z$). Let IS = (U, A) be an information system, then with any $B \subseteq A$, there is an associated equivalence relation:

$[x]_B = \mathrm{IND}_{IS}(B) = \{ (x, y) \in U \times U \mid \forall a \in B, a(x) = a(y) \}$, where $\mathrm{IND}_{IS}(B)$ is called the B-indiscernibility relation. If (x,y) satisfies the B-indiscernibility relation, we can't distinguish (separate) the x and y by the information of B.

Def. 2.6 Rough set

Let T = (U, A) and let $B \subseteq A$ and $X \subseteq U$ **,** we can approximate X using only the information contained in B by constructing the B-lower and B-upper approximations of X, denoted $U = \{ x \mid [x]_B \cap B \neq \phi \}$ and $L = \{ x \mid [x]_B \subseteq X \}$, respectively. The B-boundary region of X is defined as U-L. If U-$L \neq \phi$, then we say this set X is rough.

Def. 2.7 Rough Membership Function

The rough membership function quantifies the degree of relative overlap between the set X and the equivalence class $[x]_B$ to which x belongs. They are defined as follows:

$\mu_X^B : U \to [0,1]$ and $\mu_X^B = \dfrac{|[x]_B \cap X|}{|[x]_B|}$, here, "$|\ \ |$" is a specified measurement of relative overlap.

Consider the more general definition of approximation space, which can be used for example for similarity based on rough set model. Here, the approximation space is defined as a system $R = (U, I, v, P)$, where U is a non-empty set of objects, $I : U \to \mathbf{P}(U)$ is an uncertainty function, $v : \mathbf{P}(U) \times \mathbf{P}(U) \to [0,1]$ is a rough inclusion

function and $P : I(U) \rightarrow \{0,1\}$ is a structural function. An uncertainty function defines a neighborhood of every object x. The rough inclusion function defines the value of inclusion between two subsets of U. The following conditions where formulated for rough inclusion function:

- $v(X,X) = 1; \text{for any} X \subseteq U$,

- $v(X,Y) = 1$ implies that $v(Z,Y) \geq v(Z,X)$ for any triple $X, Y, Z \subseteq U$; if in addition $v(Y,X) = 1$ then $v(Y,Z) \geq v(X,Z)$,

- $v(\varnothing, X) = 1; \text{for any} X \subseteq U$.

In the classical definition of approximation space, we consider a pair (U, IND), where U is a non-empty set and IND is an equivalence relation on U. The classical approximation space corresponds to the approximation space $R = (U, I, v, P)$, where

* $\{I(x) : x \in U\}$ creates a partition of U $((x, y) \in IND; \text{iff} I(x) = I(y))$,

* $v(X,Y) = \dfrac{|X \cap Y|}{|X|}$ for any $X, Y \subseteq U, X \neq \varnothing$,

* $P(I(x)) = 1$; for any $x \in U$.

*

The definition of the lower *L* and the upper *U* approximations of set can be rewritten as follows:

$$L(R, X) = \{x \in U : P(I(x)) = 1, and \quad v(I(x), X) = 1\} \qquad \text{and}$$
$$U(R, X) = \{x \in U : P(I(x)) = 1, and \quad v(I(x), X) > 0\}.$$

If the uncertainty function I, defines a tolerance relation (reflexive and symmetrical relation) not being an equivalence relation then there is a variety of possibilities to define the lower and upper set approximations. Since such a set approximation is not crisp, we call it as the rough set.

3. Construction of Affinity Set

Now our initial and original ideas will be presented in this section; furthermore, we will compare these two sets of Section 2 with our affinity set

3.1. Affinity Set and Affinity

We start by presenting the meaning we give to the primitive notion of *set.* Since our objective in this section is to formalize the time-dependence of affinity between an element and a set, our meaning should encompass the variability of shape or content of a set.

Def. 3.1. By affinity set we mean any object (real or abstract) that creates affinity between object*s.*

Ex. 3.1. An institution or company is an affinity set for it is an object that creates affinity between people that make them work together.

Ex. 3.2. A political party is an affinity set for it creates affinity between people to make them work together for a program or ideal.

From the above examples we deduce that our set notion is wider that the traditional set notion. Indeed, a traditional set is a particular case of affinity set with constant content where the affinity is expressed by belongingness.

Def. 3.2. Let e and A be a subject and an affinity set, respectively. Let I be a subset of the time axis $[0, +\infty[$. The affinity between e and A is represented by a function

$$\mathrm{M}_{\mathrm{A}}^{e}(\,.\,): \mathrm{I} \rightarrow [0,1]$$

$$t \rightarrow \mathrm{M}_{\mathrm{A}}^{e}(t).$$

The value $\mathrm{M}_{\mathrm{A}}^{e}(t)$ expresses the degree of affinity between the subject e and the affinity set A at time t. In fact the affinity function is a fuzzy set defined on the subset I of the time axis. When $\mathrm{M}_{\mathrm{A}}^{e}(t)= 1$ this means that the affinity of e with the affinity set A is complete or at the maximum level at time t; *it doesn't mean that e belongs to* A*, unless the considered affinity is the belongingness*.

Def. 3.3. The universal set, denoted by U, is the affinity set representing the fundamental principle of existence. We have

$$M_U^e(\, . \,): [0,+\infty[\to [0,1]$$

$$t \to M_A^e(t)$$

and $M_U^e(t)=1$, for all existing objects at time t, and for all times t.

In other words the affinity set defined by the affinity "existence" has complete affinity with all objects that have existed in the past, that exist in the present and that will exist in the future. In general, in real world situations, some traditional referential set V, such that when an object e is not in V, $M_A^e(t)=0$ for all t in $I \subset [0,+\infty[$, can be determined., we assume that the sets V and I are given.

Def. 3.4. Let A be an affinity set. Then the function defining A is

$$F_A (., .): V \times I \to [0,1]$$

(1)

$$(e, t) \to F_A (e, t)= M_A^e(t)$$

It has to be noted also that the behavior of an affinity set A over time can be investigated through its function $F_A (., .)$.

The maximum affinity $M_A^e(t)=1$ may not be reached at any time in real-world problems. In order to consider various situations we introduce the following definition.
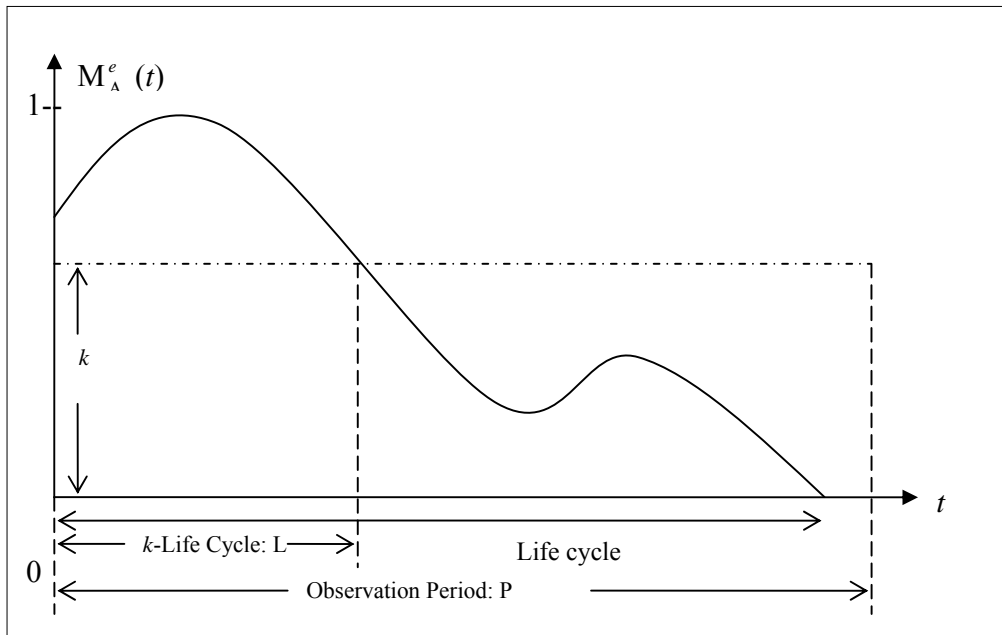
Def. 3.5. Let A be an affinity set and $k \in [0,1]$. We say that an element e is in the *t-k*-Core of the affinity set A at time t, denoted by *t-k*-Core(A), if $M_A^e(t) \geq k$, that is,

$$t - k - Core(A) = \{e | M_A^e(t) \geq k\}$$

(2)

when $k =1$, *t-k*-Core(A) is simply called the core of A at time t, denoted by *t*-Core(A).

Def. 3.6. An observation period is defined as the period (continuous or discrete) that one is interested in analyzing the behavior of an element *e* of V with respect to an affinity set A (an illustration is given in Figure 1 below).

Figure 1. Illustration of the affinity between an element *e* and an affinity set over an observation period P.

Def. 3.7. Let A be an affinity set and $k \in [0,1]$. A subset T (discrete or continuous) of I is said to be the *k*-life cycle of an element *e* with respect to A if

$$\mathrm{M}_{\mathrm{A}}^{e}(t) \geq k, \text{ for all } t \in \mathrm{T} \text{ and } \mathrm{M}_{\mathrm{A}}^{e}(t) < k, \text{ elsewhere in I.}$$

3.2. Indirect Affinity and Operations

When affinity between subjects takes place via a medium, then we say that there is indirect affinity between them. In this section we give a formal definition of the indirect affinity. The notion of harmony between objects with respect to an affinity set is also formalized.

Def. 3.8. Let A be an affinity set and $k \in [0,1]$. Let D be a subset of V. We say that there is a *k*-indirect affinity degree with respect to A, at time *t*, between the elements of D, if they all belong to the *t-k*-Core(A), that is $D \subset t - k - Core$ (A). We say that there is *k*-indirect affinity degree with respect to A, during an observation period T , between the elements of D, if $D \subset t - k - Core$ (A) at any time *t* in T.

In the following definitions 2.10-2.14, we assume that A and B are two given affinity sets defined on I and V.

Def. 3.9. We say that A and B are equal at time t if $M_A^e(t) = M_B^e(t)$, for all *e* in V. Then we write A = B at time *t*. If A and B are considered in an observation period T, then A = B during this period if $M_A^e(t) = M_B^e(t)$, for all *e* in V and all *t* in T.

Def. 3.10. We say that A is contained in B at time t if $M_A^e(t) \le M_B^e(t)$, for all *e* in V. Then we write $A \subset B$ at time *t*. In the case A and B are considered in an observation period T, then $A \subset B$ during this period if $M_A^e(t) \le M_B^e(t)$, for all *e* in V and all *t* in T.

Def. 3.11. The union of A and B at time *t*, denoted by $A \cup B$, is defined by the function $F_{A \cup B}(t, e) = M_{A \cup B}^e(t) = \text{Max}\{M_A^e(t), M_B^e(t)\}$, for all *e* in V. In the case A and B are considered in an observation period T, then during this period, $A \cup B$ is defined by the function $F_{A \cup B}(t, e) = M_{A \cup B}^e(t) = \text{Max}\{M_A^e(t), M_B^e(t)\}$, for all *e* in V and all *t* in T.

Def. 3.12. The intersection of affinity sets A and B at time *t*, denoted by $A \cap B$, is defined by the function $F_{A \cap B}(t, e) = M_{A \cap B}^e(t) = \text{Min}\{M_A^e(t), M_B^e(t)\}$, for all *e* in V. In the case A and B are considered in an observation period T, then during this period, $A \cap B$ is defined by the function $F_{A \cap B}(t, e) = M_{A \cap B}^e(t) = \text{Min}\{M_A^e(t), M_B^e(t)\}$, for all *e* in V and all *t* in T.

Def. 3.13. B is said to be the complement of A at time t if it is defined by the following function $F_B(t, e) = M_B^e(t) = 1 - M_A^e(t)$, for all e in V. In the case A an B are considered in an observation period T, then during this period, B is defined by the function $F_B(t, e) = M_B^e(t) = 1 - M_A^e(t)$, for all e in V and all t in T.

We make a simple comparison among the fuzzy set, rough set and affinity set in Table 1.

Table 1 Comparison of Affinity Set, Rough Set and Fuzzy Set

Attribute\Set	Fuzzy	Rough	Affinity
Logic	Multivalent	Bivalent/Multivalent	Bivalent/Multivalent
Classification for Set	Membership	Approximation	Affinity
Set Boundary	Subjective and Fixed	Objective and Dynamic	Subjective and Dynamic

In traditional set theory the boundary of a set is precise and we can decide whether an element is in a set. In fuzzy set theory the boundary of a set is fuzzy-fixed. The main purpose of the rough set analysis is the induction of approximations of concepts from the acquired data. The affinity set theory is quite different in the sense that the membership to an affinity set depends on time. Affinity set needs no assumptions of membership function and no approximations of sets.

4. Potential Application of Forecasting

In fact any time series method can be used to predict the behavior of any element e in V with respect to an affinity set. A based on past data, if it is possible to define the affinity set A. In this paper we propose a new forecasting method based on game theory and affinity set. Assume that an affinity set A and a universe V are given and some data are available at some past periods $t_1, t_2, ..., t_n$ on the behavior of an elements e in V with respect to the affinity set A as described in the following matrix

$$D = \begin{matrix} & \begin{matrix} t_1 & ... & t_n \end{matrix} \\ \begin{matrix} A \\ \overline{A} \end{matrix} & \begin{pmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \end{pmatrix} \end{matrix}$$

where $\overline{A}$ is the affinity set complementary to A (see Definition 3.13), the entry a_{1j} is the affinity degree of the element e with respect to the affinity set A at the period t_j and $a_{2j} = 1 - a_{1j}$ is the affinity degree of the element e with respect to the affinity set $\overline{A}$ at the same period. Here a decision maker wants to forecast the behavior of the element e at the next period t_{n+1}. Interestingly we can look at the situation as a game between the decision maker and Nature i.e. a game against Nature. Indeed, the decision maker faces an uncertain situation represented by future behavior of the element e, one way to handle the situation is to adopt the maxmin decision making under uncertainty principle [4] by considering the situation as a game against Nature [1]. Thus, the matrix D can be considered as a matrix game between the decision maker and Nature, where the decision maker is the maximizing player who chooses between A and $\overline{A}$ and Nature is the minimizing player who chooses the time periods.

Def. 4.1. A pair of strategies (i_0, j_0) where $i_0 \in \{1,2\}$ and $j_0 \in \{1,..,n\}$ is said to be Nash equilibrium [6] of the matrix game D if

$$a_{ij_0} \le a_{i_0 j_0} \le a_{i_0 j}, \quad \text{for all } i \in \{1,2\} \text{ and } j \in \{1,..,n\}.$$

Assume that the game has a Nash equilibrium (i_0, j_0). In terms of affinity, this equilibrium can be interpreted as follows. If $i_0 = 1$, the decision maker will favor the affinity of the element e with the affinity set A rather that affinity with $\overline{A}$, with affinity degree $a_{i_0 j_0}$. In the case $i_0 = 2$ the decision maker will favor the affinity of the element e with $\overline{A}$ rather than with A, with affinity degree $a_{i_0 j_0}$. It may happen that the matrix D has no Nash equilibrium in pure strategies, then the two players have to use mixed strategies. A mixed strategy for Nature is a probability distribution over the set of its pure strategies, that is, it is a vector $y = (y_1, y_2,..., y_n)$ such that

$$\sum_{j=1}^{n} y_j = 1 \text{ and } y_j \geq 0, \ j = \overline{1,n},$$

Similarly, a mixed strategy for the decision maker is a vector $x = (x_1, x_2)$ such that

$$x_1 + x_2 = 1 \text{ and } x_i \geq 0, \ i = 1,2 \ .$$

The payoffs of players become expected payoffs. The payoff of the decision maker is $x^T Dy$ and that of Nature is $-x^T Dy$. Any matrix game has always a Nash equilibrium in mixed strategies [1]. A Nash equilibrium in mixed strategies is defined by

$$x^T Dy^* \leq x^{*T} Dy^* \leq x^{*T} Dy \quad \text{for all mixed strategies } x \text{ and } y \ .$$

The mixed strategy x^* of the decision maker can be interpreted as follows. The decision maker will favor A with weight x_1 and $\overline{A}$ with the weight x_2. He can also use these two evaluations to rank the sets A and $\overline{A}$ from his point of view. The expected affinity degree of the element e in the period t_{n+1} with each of the affinity sets can be defined as $M_A^e(t_{n+1}) = \sum_{1}^{n} a_{1j} y_j^*$ and $M_{\overline{A}}^e(t_{n+1}) = \sum_{1}^{n} a_{2j} y_j^*$, respectively.

The mixed strategy y^* of Nature can be interpreted as the weights Nature assigns to the periods in order to minimize the expected affinity of the decision maker. Let us illustrate our approach by examples.

Ex. 4.1. Assume that a stock market manager wants to predict if he can sale out or buy in some stocks for his company. Assume, for simplicity, that by experience he classifies his decisions into "buy in" or "sale out". These two possible states can be considered as two affinity complementary sets A and $\overline{A}$, respectively. The decision will be the element e, and the set A is defined as the "decision of buying in". Assume that in the past four quarters the manager has recorded the following data representing the affinity degrees of "buy in" with respect to the affinity set A

t_1	t_2	t_3	t_4
0.5	0.7	0.3	0.6

The manager wants to forecast the affinity e with respect to the affinity sets A and $\overline{A}$ in the next period t_{4+1}. Then the following matrix game between the manager and Nature is presented, which is treated by the following data set.

$$D= \begin{array}{c} A \\ \overline{A} \end{array} \begin{pmatrix} 0.6 & 0.7 & 0.8 & 0.6 \\ 0.4 & 0.3 & 0.2 & 0.4 \end{pmatrix}$$

Then the optimal mixed strategies are $x = (x_1, x_2) = (1, 0)$ and $y = (y_1, y_2, y_3, y_4) = (1, 0, 0, 0)$. Hence the predicted affinities of the demand, by the manager, for the next period are $M_A^e(t_{4+1})$=0.6 and $M_{\overline{A}}^e(t_{4+1})$=0.4. In this case in the next period it is forecasted that the "buy in" decision is beneficial to this manager, because it will have greater affinity to "buy in" than to "sale out". The result x_2=0, can be explained by the fact that in the matrix D the row 1 dominates the row 2, that is, the affinity degree of A (buy in) is higher that the affinity degree of $\overline{A}$ (sale out) for each of the four considered periods.

5. Conclusions and Recommendations

In this paper we proposed a framework based on fuzzy sets for the concept of affinity, which allows its investigation by fuzzy set tools and structures. We studied one type of affinity: The indirect affinity at the beginning. It was pointed out that the indirect affinity requires a medium. We think that the investigation of affinity in social sciences using our fuzzy framework is a worthy direction of research. We hope that this paper will inspire and attract more researchers for the investigation and application of the affinity concept.

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