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Numerical Results of Convergence Rates in Regularization for Ill-Posed Mixed Variational Inequalities

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Abstract

In this note some numerical experiments to illustration for convergence rates of regularized solution for ill-posed inverse-strongly monotone mixed variational inequalities are presented.

Keywords: Monotone operators, hemi-continuous, strictly convex Banach space, Fréchet differentiable, weakly lower semicontinuous functional and Tikhonov regularization

1 Introduction

Variational inequality problems appear in many fields of applied mathematics such as convex programming, nonlinear equations, equilibrium models in economics, technics (see [2], [7]). These problems can be defined over finite-dimensional spaces as well as over infinite-dimensional spaces. In this paper, we suppose that they are defined on a real reflexive Banach space X having a property that the weak and norm convergences of any sequence in X infoly its strong convergences, and the dual space X^* of X is strictly convex. For the sake of simplicity, the norms of X and X^* are denoted by the symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Then, the mixed variational inequality problem can be formulated as follows: for a given $f \in X^*$, find an element $x_0 \in X$ such that

$$\langle A(x_0) - f, x - x_0\rangle + \varphi(x) - \varphi(x_0) \geq 0, \quad \forall x \in X. \tag{1}$$

where A is a hemi-continuous and monotone operator from X into X^*, and $\varphi(x)$ is an weakly lower semicontinuous and proper convex functional on X. We will suppose that problem (1) has at least one solution. For existence theorems, we refer the reader to [5]. Many problems can be seen as special cases of the problem (1). When φ is the indicator function of a closed convex set K in X, that is

$$\varphi(x) = I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

then the problem (1) is equivalent to that of finding $x_0 \in K$ such that

$$\langle A(x_0) - f, x - x_0 \rangle \geq 0, \quad \forall x \in K.$$

When K is the whole space X, this variational inequality is of the form of operator equation $A(x) = f$. When A is the Gâteaux derivative of a finite-valued convex function F defined on X, Problem (1) becomes the nondifferentiable convex optimization problem (see [5]):

$$\min_{x \in X} F(x) + \varphi(x). \tag{2}$$

The problem (1) is in general ill-posed. By ill-posedness we mean that solutions do not depend continuously on the data (A, f, φ). Many methods have been proposed for solving Problem (1), for example the proximal point method [10], the auxiliary problem method [6] and the regularization method... The last introduced by Liskoves for solving mixed variational inequality problems (see [8]) by using the following mixed variational inequality

$$\langle A_h(x_\alpha^\tau) + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x - x_\alpha^\tau \rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x_\alpha^\tau) \geq 0, \quad \forall x \in X, \tag{3}$$

where α is regularization parameter, U^s is a generalized duality mapping of X, i.e., U^s is a mapping from X onto X^* satisfying $\langle U^s(x), x \rangle = \|x\|^s,\ \|U^s(x)\| = \|x\|^{s-1},\ s \geq 2$, $(A_h, f_\delta, \varphi_\varepsilon)$ is approximation of (A, f, φ), $\tau = (h, \delta, \varepsilon)$ and x_* is in X playing the role of a criterion of selection. By the choice of x_*, we can obtain approximate solutions.

In [8] it was shown that the existence and uniqueness of the solution x_α^τ for every $\alpha > 0$. The regularized solution x_α^τ converges to $x_0 \in S_0$, where S_0 is the set of solutions of (1) which is assumed to be nonempty with x_*-minimum norm solution, i.e.,

$$\|x_0 - x_*\| = \min_{x \in S_0} \|x - x_*\|,$$

if $(h + \delta + \varepsilon)/\alpha, \alpha \to 0$.

In this paper, we use inequality (3) with the following conditions posed on the perturbations: $A_h : X \to X^*$ is the hemi-continuous monotone operator approximated A in the sense

$$\|A_h(x) - A(x)\| \leq hg(\|x\|), \quad h \to 0$$

where $g(t)$ is a nonegative function satisfying the condition $g(t) \leq g_0 + g_1 t^\eta,\ \eta = s-1,\ g_0, g_1 \geq 0,\ f_\delta$ are approximations of $f : \|f_\delta - f\| \leq \delta,\ \delta \to 0,\ \varphi_\varepsilon$ are functionals defined on X to be of the same properties as φ, and

$$\begin{aligned} |\varphi(x) - \varphi_\varepsilon(x)| &\leq \varepsilon d(\|x\|), \quad \varepsilon \to 0, \\ |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| &\leq C_0\|x - y\|, \quad \forall x, y \in X, \end{aligned} \tag{4}$$

for some positive constant C_0 and $d(t)$ has the same properties as $g(t)$.
We also assume that the dual mapping U^s satisfies the following conditions

$$\begin{aligned} \langle U^s(x) - U^s(y), x - y\rangle &\geq m_s\|x - y\|^s, \quad m_s > 0, \\ \|U^s(x) - U^s(y)\| &\leq C(R)\|x - y\|^\nu, \quad 0 < \nu \leq 1, \end{aligned}$$

where $C(R)$, $R > 0$ is a positive increasing function on $R = \max\{\|x\|,\ \|y\|\}$. It is well-known that when $X = L^2[a, b]$ is a Hilbert space, then $U^s = I$, $s = 2$, $m_s = 1$, $\nu = 1$ and $C(R) = 1$, where I denotes the identity operator in the setting space (see [1], [12]).

The problem of choosing the value of the regularization parameter α depending on τ, i.e., $\alpha = \alpha(h, \delta, \varepsilon)$, and the convergence rate for the regularized solution x^τ_α are studied in [3]. We show that the parameter α can be chosen by solving the equation

$$\rho(\alpha) = (h + \delta + \varepsilon)^p \alpha^{-q}, \quad p, q > 0, \tag{5}$$

where $\rho(\alpha) = \alpha\|x^\tau_\alpha - x_*\|^{s-1}$.

The finite-dimensional approximation for (3) is the important problem. We approximate (3) by the sequence of finite-dimensional problems in [4]

$$\begin{aligned} \langle A^n_h(x^\tau_{\alpha,n}) + \alpha U^{sn}(x^\tau_{\alpha,n} - x^n_*) - f^n_\delta, x^n - x^\tau_{\alpha,n}\rangle& \\ + \varphi_\varepsilon(x^n) - \varphi_\varepsilon(x^\tau_{\alpha,n}) \geq 0, \quad \forall x^n \in X_n&, \end{aligned} \tag{6}$$

where $A^n_h = P^*_n A_h P_n$, $U^{sn} = P^*_n U^s P_n$, $x^n_* = P_n x_*$, $f^n_\delta = P^*_n f_\delta$, $P_n : X \longrightarrow X_n$ is a linear projection from X onto X_n, the finite-dimensional subspace of X, P^*_n is the conjugate of P_n, and

$$X_n \subset X_{n+1}, \quad \forall n \qquad ; \quad P_n x \longrightarrow x, \quad \forall x \in X, \quad \|P_n\| = 1.$$

As also for (3), variational inequality (6) has a unique solution $x^\tau_{\alpha,n}$ for every fixed $h, \delta, \varepsilon, \alpha > 0$ and n. In [4] we also consider the modified generalized discrepancy principle for selecting $\tilde{\alpha} = \alpha(h, \delta, \varepsilon, n)$ so that $x^\tau_{\tilde{\alpha},n}$ converges to x_0 as $h, \delta, \varepsilon \longrightarrow 0$ and $n \longrightarrow \infty$ in connection with the finite-dimensional and obtain the rates of convergence for the regularized solutions in this case.

2 The obtained results

Assumption 2.1. There exists a number $\tilde{\tau} > 0$ such that

$$\|A(y) - A(x) - A'(x)(y - x)\| \leq \tilde{\tau}\|A(y) - A(x)\|, \quad \forall x \in S_0,$$

for y belonging to some neighbourhood of S_0, in which $A'(x)$ denotes the Fréchet derivative of A at x.

Rule 2.1. (see [4]) Choose $\tilde{\alpha} = \alpha(h, \delta, \varepsilon, n) \geq \alpha_0 := (c_1 h + c_2\delta + c_3\varepsilon + c_4\gamma_n)^p$, $c_i > 1, i = 1,\ 2,\ 3,\ 4$ and $0 < p < 1$ such that the following inequalities

$$\tilde{\alpha}^{1+q}\|x^\tau_{\tilde{\alpha},n} - x^n_*\|^{s-1} \geq (h + \delta + \varepsilon)^p,$$
$$\tilde{\alpha}^{1+q}\|x^\tau_{\tilde{\alpha},n} - x^n_*\|^{s-1} \leq K_1(h + \delta + \varepsilon)^p, \quad K_1 \geq 1,$$

hold.
The results on the convergence rate for regularized solution are studied in [3] as follows.

Theorem 2.1. *Assume that the following conditions hold:*
(i) A is an inverse-strongly monotone operator from X into X^*, i.e.

$$\langle A(x) - A(y), x - y\rangle \geq m_A\|A(x) - A(y)\|^2, \quad \forall x, y \in X,\ m_A > 0,$$

Fréchet differentiable in some neighbourhood of S_0, and satisfies Assumption 2.1 at $x = x_0$;
(ii) There exists an element $z \in X$ such that $A'(x_0)^*z = U^s(x_0 - x_*)$;
(iii) The parameter α is chosen by (5).
Then, we have

$$\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\| = O((h + \delta + \varepsilon)^{\mu_1}), \quad \mu_1 = \frac{1}{1+q}\min\left\{\frac{1+q-p}{s}, \frac{p}{2s}\right\}.$$

Remark 2.1. If α is chosen so that $\alpha \sim (h + \delta + \varepsilon)^\eta$, $0 < \eta < 1$, then

$$\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\| = O((h + \delta + \varepsilon)^{\mu_2}), \quad \mu_2 = \min\left\{\frac{1-\eta}{s}, \frac{\eta}{2s}\right\}.$$

And now, we consider the convergence and convergence rate for the sequence $\{x^\tau_{\tilde{\alpha},n}\}$ in [4].

Theorem 2.2. *Assume that the following conditions hold:*
(i) conditions (i), (ii) of Theorem 2.1;
(ii) the parameter $\tilde{\alpha} = \alpha(h, \delta, \varepsilon, n)$ is chosen by the Rule 2.1.
Then, we have

$$\|x^\tau_{\tilde{\alpha},n} - x_0\| = O((h + \delta + \varepsilon + \gamma_n)^{\mu_3} + \gamma_n^{\mu_4}),$$
$$\mu_3 = \min\left\{\frac{1-p}{s}, \frac{p}{2s(1+q)}\right\}, \quad \mu_4 = \min\left\{\frac{1}{s}, \frac{\nu}{s-1}\right\}.$$

Remark 2.2. If $\tilde{\alpha}$ is chosen a priory such that $\tilde{\alpha} \sim (h+\delta+\varepsilon+\gamma_n)^\eta$, $0 < \eta < 1$, then

$$\|x^\tau_{\tilde{\alpha},n} - x_0\| = O((\gamma_n^{\mu_4} + (h+\delta+\varepsilon+\gamma_n)^{\mu_5}), \quad \mu_5 = \min\left\{\frac{1-\eta}{s}, \frac{\eta}{2s}\right\}.$$

3 Numerical example

We now apply the obtained results of the previous sections to solve the following optimization problem:

$$\min_{x\in H} F(x) + \varphi(x) \tag{7}$$

with $F(x) = \frac{1}{2}\langle Ax, x\rangle$, where A is a self-adjoint linear bounded operator on a real Hilbert space H such that $\langle Ax, x\rangle \geq 0$, $\forall x \in H$. Because of the fact that $F'(x) = Ax$, x_0 is a solution of Problem (7) if and only if x_0 is a solution of the following problem (see [5])

$$\langle A(x_0), x - x_0\rangle + \varphi(x) - \varphi(x_0) \geq 0, \quad \forall x \in H.$$

This is Problem (1) with $f = \theta \in H$.

Obviously, $A : H \to H$ is an inverse-strongly monotone operator (see [9]) and Fréchet differentiable with the Fréchet derivative A. In this case condition (ii) of Theorem 2.1 is described by

$$A(x_0)^* z = x_0, \quad (x_* = \theta).$$

The last equation holds if $A(x_0)^*$ satisfies coerciveness in H.

Consider the case φ is nonsmooth. It can be approximated by a sequence of smooth and monotone functions φ_ε (see [13]). So making use of the method of regularization (3) to be of the form

$$A_h(x^\tau_\alpha) + \alpha I(x^\tau_\alpha - x_*) + \varphi'_\varepsilon(x^\tau_\alpha) = f_\delta \tag{8}$$

we can find an approximation solution for (7), where $\alpha > 0$ is sufficiently small.

The computational results here are obtained by using MATLAB. We shall show this by examples.

3.1. Consider the case when $H = \mathbb{R}^M$, with

• A is a $M \times M$ matrix defined by $A = B^T B$, with $B = (b_{ij})^M_{i,j=1}$,

$$b_{1j} = sin(1), \quad j = 1, ..., M$$
$$b_{2j} = 2sin(1), \quad j = 1, ..., M$$
$$b_{ij} = cos(i)sin(j), \quad i = 3, ..., M, \quad j = 1, ..., M.$$

$A_h = Ih + A$, where I denotes the identity matrix.

- $f_\delta = (\delta, 0, ..., 0)^T \in \mathbb{R}^M$ is an approximation of $f = (0, ..., 0)^T \in \mathbb{R}^M$.
- The function $\varphi : \mathbb{R}^M \to \mathbb{R}$ is chosen as follows

$$\varphi(x) = \begin{cases} 0 & , \quad x_1 \le 0, \\ x_1 & , \quad x_1 > 0, \end{cases}$$

where $x = (x_1, x_2, ..., x_M) \in \mathbb{R}^M$. Then φ is a proper convex continuous, but it is not a differentiable function at $x = (0, x_2, ..., x_M)$. We can approximate φ by φ_ε:

$$\varphi_\varepsilon(x) = \begin{cases} 0 & , \quad x_1 \le -\varepsilon, \\ \dfrac{(x_1+\varepsilon)^2}{4\varepsilon} & , \quad -\varepsilon < x_1 \le \varepsilon, \\ x_1 & , \quad x_1 > \varepsilon. \end{cases}$$

This function is nonegative, differentiable for every $\varepsilon > 0$. It is easy to see that φ satisfies the first of conditions (4). In the other hand, we have φ'_ε is a monotone operator from $\mathbb{R}^M$ into $\mathbb{R}^M$ with

$$\varphi'_\varepsilon(x) = \begin{cases} (0, 0, ..., 0) & , \quad x_1 \le -\varepsilon, \\ \dfrac{1}{2\varepsilon}(x_1 + \varepsilon, 0, ..., 0) & , \quad -\varepsilon < x_1 \le \varepsilon, \\ (1, 0, ..., 0) & , \quad x_1 > \varepsilon, \end{cases}$$

and the second of conditions (4) is satisfied.

It is clear that, $x_0 = (0, 0, ..., 0)^T \in \mathbb{R}^M$ is a solution to the problem (7) with minimal norm. Apply Theorem 2.1 for $h = \delta = \varepsilon = \dfrac{1}{M^2}$ and $\alpha \sim (h + \delta + \varepsilon)^{2/3}$, we should obtain the convergence rates $r^\tau_{\alpha,M} = \|x^\tau_{\alpha,M} - x_0\|$. Taking account of the iterative method in [14] for finding regularized solutions, with the following stopping criterion: if

$$\max_{1 \le j \le M} |x_j^{(m)} - x_j^{(m-1)}| \le 10^{-5}$$

then stop. We get the following tables of computational results:

M	α	$r^\tau_{\alpha,M}$
4	0.15749	0.011187
8	0.0625	0.0070197
16	0.024803	0.0027643
32	0.0098431	0.00089026
64	0.0039063	0.00044951

Table 2.1

3.2. Consider the case $H = L^2[0,1]$, with
• $A : L^2[0,1] \to L^2[0,1]$ is difined by

$$(Ax)(t) = \int_0^1 k_i(t,s)x(s)ds, \quad i = 1,2,$$

where

$$k_1(t,s) = \begin{cases} t(1-s) & , \quad \text{if} \quad t \le s, \\ s(1-t) & , \quad \text{if} \quad s < t, \end{cases}$$

and

$$k_2(t,s) = \begin{cases} \dfrac{(1-s)^2st^2}{2} - \dfrac{(1-s)^2t^3(1+2s)}{6} + \\ \qquad + \dfrac{(t-s)^3}{6}, & \text{if} \quad t \ge s, \\ \dfrac{s^2(1-s)(1-t)^2}{2} + \dfrac{s^2(1-t)^3(2s-3)}{6} + \\ \qquad + \dfrac{(s-t)^3}{6}, & \text{if} \quad t < s, \end{cases}$$

are kernel functionals which are difined on the square $\{0 \le t,s \le 1\}$. Obviously, A is nonegative, self-adjoint and completely continuous linear operator on $L^2[0,1]$.

$$(A_hx)(t) = \int_0^1 k_{ih}(t,s)x(s)ds, \quad i = 1,2$$

is an approximation of A, where $k_{ih}(t,s) = k_i(t,s) + h(t,s)$, with $|h(t,s)| \le h$, $\forall\, t,s$ and $h \to +0$.
• The function $\varphi : L^2[0,1] \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi(x) = F_1\big(\frac{1}{2}\langle Ax, x\rangle\big),$$

with $F_1 : \mathbb{R} \to \mathbb{R}$ is chosen as follows

$$F_1(t) = \begin{cases} 0 & , \quad t \le a_0, \\ c(t-a_0) & , \quad t > a_0, \ c, a_0 > 0. \end{cases}$$

The function $\varphi_\varepsilon(x) = F_{1\varepsilon}\big(\frac{1}{2}\langle Ax, x\rangle\big)$ is an approximation of $\varphi(x)$ with

$$F_{1\varepsilon}(t) = \begin{cases} 0 & , \quad t \le a_0, \\ \dfrac{c}{1+\varepsilon}(t-a_0)^{1+\varepsilon} & , \quad t > a_0. \end{cases}$$

Obviously, φ_ε satifies the conditons in (4) and $\varphi'_\varepsilon(x) = F'_{1\varepsilon}\big(\frac{1}{2}\langle Ax, x\rangle\big)Ax$ is an monotone operator from $L^2[0,1]$ to $L^2[0,1]$.

- $f_\delta(t) = \delta$, $t \in [0,1]$ is an approximation of $f = \theta \in L^2[0,1]$.

We compute the regularized solutions $x^\tau_{\alpha,n}$ by approximating $L^2[0,1]$ by the sequence of the linear subspaces H_n which is a set of all linear combinations of $\{\phi_1, \phi_2, ..., \phi_n\}$ defined on uniform grid of $n+1$ points in $[0,1]$:

$$\phi_j(t) = \begin{cases} 1 & , \quad t \in (t_{j-1}, t_j] \\ 0 & , \quad t \notin (t_{j-1}, t_j] \end{cases}$$

Where

$$P_n x(t) = \sum_{j=1}^{n} x(t_j)\phi_j(t),$$

with $\|P_n\| = 1$ and $\|(I - P_n)x_0\| = O(n^{-1})$, $\forall x \in L^2[0,1]$ (see [11]). Then, finite-dimensional regularized equation (8) has form

$$B_h \tilde{x} + \varphi_\varepsilon^{'n}(\tilde{x}) = f_\delta^n, \tag{9}$$

where

$$B_h = \begin{pmatrix} b_1 k_{ih}(t_1,t_1) + \alpha & b_2 k_{ih}(t_1,t_2) & ... & b_n k_{ih}(t_1,t_n) \\ b_1 k_{ih}(t_2,t_1) & b_2 k_{ih}(t_2,t_2) + \alpha & ... & b_n k_{ih}(t_2,t_n) \\ ... & ... & ... & ... \\ b_1 k_{ih}(t_n,t_1) & b_2 k_{ih}(t_n,t_2) & ... & b_n k_{ih}(t_n,t_n) + \alpha \end{pmatrix}$$

$$b_1 = b_2 = ... = b_{n-1} = \frac{1}{n},\ b_n = \frac{1}{2n},$$
$$\varphi_\varepsilon^{'n}(\tilde{x}) = (\varphi'_\varepsilon(\tilde{x}_1), ..., \varphi'_\varepsilon(\tilde{x}_n))^T, \quad f_\delta^n = (\delta, ..., \delta)^T,$$
$$\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)^T, \quad \tilde{x}_j \sim x(t_j),\ j = 1, ..., n.$$

Apply Theorem 2.2 for $\tilde{\alpha} \sim (h + \delta + \varepsilon + \gamma_n)^\eta$, $0 < \eta < 1$, we should obtain the convergence rates $r^\tau_{\tilde{\alpha},n} = \|x^\tau_{\tilde{\alpha},n} - x_0\|$. Taking account of the iterative method in [14] for finding regularized solutions, with the following stopping criterion: if $\max_{1\le j\le n} |x_j^{(m)} - x_j^{(m-1)}| \le 10^{-5}$ then stop. We get the tables of computational results with $c = 1/2$, $a_0 = 10^{-3}$.

The numerical results for different two problems are presented in the following tables. The problems 1, 2 are respectively sudied to the functions $k_1(t,s)$, $F_1(t)$ and $k_2(t,s)$, $F_1(t)$.

Problem	n	$\tilde{\alpha}$	$r^\tau_{\tilde{\alpha},n}$
1	40	0.15811	0.073584
2	40	0.15811	0.085569
1	100	0.1	0.032054
2	100	0.1	0.038583

Table 2.2: $\eta = \frac{1}{2}$, $\delta = h = \varepsilon = \frac{1}{n}$

Problem	n	$\tilde{\alpha}$	$r^{\tau}_{\tilde{\alpha},n}$
1	40	0.15811	0.0012529
2	40	0.15811	0.0013798
1	100	0.1	0.00036562
2	100	0.1	0.0004154

Table 2.3: $\eta = \dfrac{1}{2}$, $\delta = h = \varepsilon = \dfrac{1}{n^2}$

From the numerical tables mentioned above we have the following remarks
- For h, δ, ε to be small, approximate solutions are near to the exact solution of the original problem;
- The convergence rates of regularized solutions depend on the choice of values of α depending on h, δ, ε.

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