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# A Moments Approach to Option Valuation Models

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**Abstract**

This paper presents an approach for computing option valuation models as a function of the underlying distribution's first two moments about zero. The moments approach is a useful aid in computing closed-form expressions of option valuations when the underlying distribution exhibits mean-reversion, seasonality, and other non-standard features. This approach is applied to the case when the underlying asset terminal distribution follows either the lognormal or the normal distribution. The moments approach is particularly useful for real options where attention is often focused on identifying the appropriate underlying distribution to model.

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**Keywords**: Option pricing, lognormal distribution, normal distribution, stochastic processes

## 1 Introduction

The objective here is to provide an alternative moments approach for developing option valuation models within two single factor frameworks. These two single factor frameworks are based on the reference instrument following either a lognormal distribution or a normal distribution. The reference instrument's stochastic differential equation is assumed linear in the instrument, although not necessarily linear in time. The reference index does not have to be a marketable asset.

The goal here is to introduce an alternative representation of well-known results. This alternative representation is particularly useful in situations where the focus is on estimating the underlying distribution rather than creating the appropriate self-financing, dynamic replicating portfolio. For example, if the objective is to estimate the value of an option based on temperature at a particular location, the focus is typically more on meteorology than particular hedging strategies. Another example is an energy option where the underlying energy source is known to exhibit highly seasonal prices.

The paper proceeds as follows: Section II presents two theorems for the expected terminal value of options where the underlying instrument follows either lognormal or normal distribution. Section III presents the key general linear theorem of Mikosch [8] with a particular focus on option valuation. Section IV illustrates a variety of option valuation models based on the moments approach.

## 2 Expected Terminal Values

In this section, the expected terminal values for call and put options are given for the lognormal distribution in Theorem 1 and for the normal distribution in Theorem 2. The implications of these results are illustrated with practical examples in Section IV. The objective here is not to resolve whether it is appropriate to take the expectation under the equivalent martingale measure (risk neutral probability measure) and discount at the risk free interest rate or whether some other expectation should be taken and discounted at an alternative interest rate. The focus is on finding the appropriate expected terminal value from which option values and risk measures can be computed. The usefulness of the following two theorems will be illustrated particularly in Section IV.

**Theorem 1. Expected Terminal Option Values - Lognormal Distribution**
Assuming the terminal value of the underlying reference index, $S_t$, is distributed lognormal, then[1]

$$E_0[C_t] = E_0[S_t]N(d_1) - XN(d_1 - B_1) \quad (1a)$$
$$E_0[P_t] = XN(-d_1 + B_1) - E_0[S_t]N(-d_1) \quad (1b)$$

where

$$B_1 = \left[\ln\left\{\frac{E_0[S_t^2]}{E_0[S_t]^2}\right\}\right]^{1/2} \quad (1c)$$
$$d_1 = \frac{\ln\{E_0[S_t]/X\}}{B_1} + \frac{B_1}{2} \quad (1d)$$

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[1] Proof available upon request.

$$N(d) = \int_{-\infty}^{d} \frac{\exp\{-\eta^2/2\}}{\sqrt{2\pi}} d\eta \quad (1e)$$

If we assume the underlying reference index grows at the continuously compounded risk-free rate $r$, then $\mu_1 = \ln(S_t) + rt - \frac{\sigma^2 t}{2}$ and $B_1 = \sigma_1 = \sigma\sqrt{t}$. In standard option valuation formulas, the mean and the variance are expressed in continuously compounded, annualized terms. Note that equation (1) applied here is the well-known Black-Scholes-Merton option pricing model once these terminal values are discounted.

Therefore, if the underlying reference index follows a lognormal distribution, then we could apply Theorem 1 to estimate the expected terminal option value. The focus is on estimating the first two moments of the underlying reference index distribution. We now illustrate this framework with the normal distribution.

**Theorem 2. Expected Terminal Option Values - Normal Distribution**
Assuming the terminal value of the underlying reference index, $S_t$, is distributed normal, then[2]

$$E_0[C_t] = [E(S_t) - X]N(d_n) + V(S_t)^{1/2} n(d_n) \quad (2a)$$

$$E_0[P_t] = [X - E[S_t]]N(-d_n) + V(S_t)^{1/2} n(-d_n) \quad (2b)$$

where

$$d_n = \frac{E[S_t] - X}{V(S_t)^{1/2}} \quad (2c)$$

$$V(S_t) = E[S_t^2] - E[S_t]^2 \quad (2d)$$

With these two representations of the terminal expected value for options, we turn now to general solutions for two classes of stochastic differential equations. These first two theorem help focus attention on estimation of moments around zero. Section III provides a straightforward framework to compute these moments.

## 3 Moment Estimation From Stochastic Differential Equations

The usual finance assumptions are made (see, for example, Harrison and Kreps [4]

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2 Proof available upon request.

and Harrison and Pliska [5]).[3] Two categories of Itô processes are considered, linear lognormal models and linear normal models. The goal here is to express the stochastic integral or stochastic differential equation in a form that leads to inferences about the distribution implied for the underlying instrument. Mikosch [8] identifies the following useful theorem.

**Theorem 3. General Linear Equation**
Consider the stochastic integral,

$$S_t = S_0 + \int_{\tau=0}^{\tau=t} [\mu_1(\tau)S_\tau + \mu_2(\tau)]d\tau + \int_{\tau=0}^{\tau=t} [\sigma_1(\tau)S_\tau + \sigma_2(\tau)]dz_\tau \tag{3a}$$

or the stochastic differential form

$$dS_t = [\mu_1(t)S_t + \mu_2(t)]dt + [\sigma_1(t)S_t + \sigma_2(t)]dz_t \tag{3b}$$

Let

$$y_t \equiv \exp\left\{ \int_{\tau=0}^{\tau=t} \left[ \mu_1(\tau) - \frac{\sigma_1^2(\tau)}{2} \right] d\tau + \int_{\tau=0}^{\tau=t} \sigma_1(\tau) dz_\tau \right\} \tag{3c}$$

then

$$S_t = y_t \left( S_0 + \int_{\tau=0}^{\tau=t} \frac{[\mu_2(\tau) - \sigma_1(\tau)\sigma_2(\tau)]}{y_\tau} d\tau + \int_{\tau=0}^{\tau=t} \frac{\sigma_2(\tau)}{y_\tau} dz_\tau \right) \tag{3d}$$

This theorem is applied in two cases related to option valuation, linear lognormal, and linear normal stochastic processes. Almost all single factor finance applications can be classified as one of these two cases.

## 3.1 Linear Lognormal Stochastic Processes

An Itô process is classified as a linear lognormal model if it can be represented in stochastic integral form as

$$S_t = S_0 + \int_{\tau=0}^{\tau=t} [\mu_1(\tau)S_\tau + \mu_2(\tau)]d\tau + \int_{\tau=0}^{\tau=t} \sigma_1(\tau)S_\tau dz_\tau \tag{4a}$$

or in stochastic differential form as

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[3] Available upon request.

$$dS_t = [\mu_1(t)S_t + \mu_2(t)]dt + \sigma_1(t)S_t dz_t \tag{4b}$$

where the drift term has two components, one a linear function of the underlying asset ($\mu_1(t)$) and the other independent of the asset value ($\mu_2(t)$). Note that the noise term is a linear function of the underlying asset and can be a function of time. The underlying asset for any $\tau > 0$ follows a lognormal distribution.

Linear lognormal models have a wide variety of applications in finance, a selected few are identified here:

Model 1: $dS = \mu S dt + \sigma S dz$ (Geometric Brownian Motion)

The model originally used by Black and Scholes [2] and Merton [7]. Also used in Rendleman and Bartter's [12] interest rate model.

Model 2: $dS_t = \alpha(L_t - S_t)dt + \sigma S_t dz_t$ (Mean Reversion)

This model was suggested by Pilipović [10] for modeling the mean reversion in energy prices. She also has a two factor model where $L_t$ is assumed to follow geometric Brownian motion. Multifactor models are the subject of another paper.

Model 3:
$dS_t = \{\mu S_t^{Und} - [2\pi\beta_A \sin(2\pi(t - t_A)) + 4\pi\beta_{SA} \sin(4\pi(t - t_{SA}))]\}dt + \sigma S_t^{Und} dz_t$

Also suggested by Pilipović [10] to handle deterministic seasonality.

Based on Theorem 1 and equation (3) where $\sigma_2(\tau) = 0$, let

$$y_t \equiv \exp\left\{\int_{\tau=0}^{\tau=t}\left[\mu_1(\tau) - \frac{\sigma_1^2(\tau)}{2}\right]d\tau + \int_{\tau=0}^{\tau=t}\sigma_1(\tau)dz_\tau\right\} \tag{5}$$

then

$$S_t = y_t\left(S_0 + \int_{\tau=0}^{\tau=t}\frac{\mu_2(\tau)}{y_\tau}d\tau\right) \tag{6}$$

Substituting equation (5) into (6) and rearranging, we have

$$S_t = S_0 A(0,t)\exp\left\{\int_{\tau=0}^{\tau=t}\sigma_1(\tau)dz_\tau\right\} + \int_{\tau=0}^{\tau=t}\mu_2(\tau)A(\tau,t)\exp\left\{\int_{\tau'=\tau}^{\tau'=t}\sigma_1(\tau')dz_{\tau'}\right\}d\tau \tag{7}$$

where

$$A(a,b) \equiv \exp\left\{\int_{\tau=a}^{\tau=b}\left(\mu_1(\tau) - \frac{\sigma_1^2(\tau)}{2}\right)d\tau\right\} \tag{8}$$

**Theorem 4. Linear Lognormal Stochastic Process Moments**

Assuming the future value of the reference index follows a linear lognormal stochastic process, the first two moments can be expressed as:[4]

$$E[S_t] = S_0 A(0,t)\exp\left\{\frac{\int_{\tau=0}^{\tau=t}\sigma_1^2(\tau)d\tau}{2}\right\} + \int_{\tau=0}^{\tau=t}\mu_2(\tau)A(\tau,t)\exp\left\{\frac{\int_{\tau'=\tau}^{\tau'=t}\sigma_1^2(\tau')d\tau'}{2}\right\}d\tau \quad (9)$$

$$E[S_t^2] = S_0^2 A^2(0,t)\exp\left\{2\int_{\tau=0}^{\tau=t}\sigma_1^2(\tau)d\tau\right\}$$

$$+ 2S_0 A(0,t)\int_{\tau=0}^{\tau=t}\mu_2(\tau)A(\tau,t)\exp\left\{2\int_{\tau'=\tau}^{\tau'=t}\sigma_1^2(\tau')d\tau' + \frac{\int_{\tau'=0}^{\tau'=\tau}\sigma_1^2(\tau')d\tau'}{2}\right\}d\tau$$

$$+ \int_{\tau=0}^{\tau=t}\int_{\hat{\tau}=0}^{\hat{\tau}=\tau}\mu_2(\tau)A(\tau,t)\mu_2(\hat{\tau})A(\hat{\tau},t)\exp\left\{2\int_{\tau'=\tau}^{\tau'=t}\sigma_1^2(\tau')d\tau' + \frac{\int_{\tau'=\hat{\tau}}^{\tau'=\tau}\sigma_1^2(\tau')d\tau'}{2}\right\}d\hat{\tau}d\tau$$

$$+ \int_{\tau=0}^{\tau=t}\int_{\hat{\tau}=\tau}^{\hat{\tau}=t}\mu_2(\tau)A(\tau,t)\mu_2(\hat{\tau})A(\hat{\tau},t)\exp\left\{2\int_{\tau'=\hat{\tau}}^{\tau'=t}\sigma_1^2(\tau')d\tau' + \frac{\int_{\tau'=\tau}^{\tau'=\hat{\tau}}\sigma_1^2(\tau')d\tau'}{2}\right\}d\hat{\tau}d\tau \quad (10)$$

Although the first two moments are rather messy, for most finance applications, these two moments can be reduced to rather simple expressions as illustrated in Section IV.

### 3.2 Linear Normal Stochastic Processes

An Itô process is classified as a linear normal model if it can be represented in stochastic integral form as

$$S_t = S_0 + \int_{\tau=0}^{\tau=t}[\mu_1(\tau)S_\tau + \mu_2(\tau)]d\tau + \int_{\tau=0}^{\tau=t}\sigma_2(\tau)dz_\tau \quad (11a)$$

or in stochastic differential equation (SDE) form as

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[4] Proof available upon request.

$$dS_t = [\mu_1(t)S_t + \mu_2(t)]dt + \sigma_2(t)dz_t \tag{11b}$$

where the drift term has two components, one a linear function of the underlying asset ($\mu_1(t)$) and the other independent of the asset value ($\mu_2(t)$). Note that the noise term is a linear function of the underlying asset and can be a function of time. The underlying asset for any $\tau > 0$ follows a lognormal distribution.

Linear normal models have a wide variety of applications in finance; a few are listed briefly here.

Model 1: $dS = \mu dt + \sigma dz$ (Arithmetic Brownian motion with arithmetic drift)
An arithmetic Brownian motion model used in various forms by Bachelier [1], Murphy [9], and Johnson and Barz [6]. See also Smith [14].

Model 2: $dS = \mu S dt + \sigma dz$ (Arithmetic Brownian motion with geometric drift)
Used by Poitras [11] to model spreads. Also found in Cox and Ross [3], equation (10), p. 151.

Model 3: $dS = \kappa(\alpha - S)dt + \sigma dz$ (Mean reverting arithmetic Brownian motion)
A mean reverting model used by Schwartz [13], where $\kappa > 0$. This model has also been used by Vasicek's model for interest rates follows this generic form.

Based on Theorem 3 and equation (3c) where $\sigma_1(\tau) = 0$, let

$$y_t \equiv \exp\left\{\int_{\tau=0}^{\tau=t} \mu_1(\tau)d\tau\right\} \tag{12}$$

then

$$S_t = y_t\left(S_0 + \int_{\tau=0}^{\tau=t} \frac{\mu_2(\tau)}{y_\tau}d\tau + \int_{\tau=0}^{\tau=t} \frac{\sigma_2(\tau)}{y_\tau}dz_\tau\right) \tag{13}$$

**Theorem 5. Linear Normal Stochastic Process Moments**
Assuming the future value of the reference index follows a linear normal stochastic process, the first two moments can be expressed as:[5]

$$E[S_t] = S_0 \exp\left\{\int_{\tau=0}^{\tau=t} \mu_1(\tau)d\tau\right\} + \int_{\tau=0}^{\tau=t} \mu_2(\tau)\exp\left\{\int_{\tau'=\tau}^{\tau'=t} \mu_1(\tau')d\tau'\right\}d\tau \tag{14}$$

$$E[S_t^2] = E[S_t]^2 + \int_{\tau=0}^{\tau=t} \sigma_2^2(\tau)\exp\left\{2\int_{\tau'=\tau}^{\tau'=t} \mu_1(\tau')d\tau'\right\}d\tau \tag{15}$$

Although the first two moments appear complex, for most finance applications, these two moments can be reduced to rather simple expressions as illustrated in Section IV.

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5 Proof available upon request.

# 4 Illustrations of the Moments Option Valuation Model

## 4.1 Lognormal Distribution Applications

The results given in Theorem 1 and 4 above are illustrated with the following two applications, geometric Brownian motion with geometric drift and geometric Brownian motion with mean reversion.

*Geometric Brownian Motion with Geometric Drift*
Consider $\mu_1(\tau)=\mu;\mu_2(\tau)=0;\sigma_1(\tau)=\sigma;\sigma_2(\tau)=0$, then based on Theorem 3

$$S_t = S_0 + \int_{\tau=0}^{\tau=t} \mu S_\tau d\tau + \int_{\tau=0}^{\tau=t} \sigma S_\tau dz_\tau \tag{16a}$$

or

$$dS_t = \mu S_t dt + \sigma S_t dz_t \tag{16b}$$

which is the well-known geometric Brownian motion (GBM). From equation (3c),

$$\begin{aligned} y_t &\equiv \exp\left\{ \int_{\tau=0}^{\tau=t} \left[ \mu - \frac{\sigma^2}{2} \right] d\tau + \sigma \int_{\tau=0}^{\tau=t} dz_\tau \right\} \\ &= \exp\left\{ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \int_{\tau=0}^{\tau=t} dz_\tau \right\} \end{aligned} \tag{17}$$

and thus from equation (3d)

$$S_t = y_t(S_0) = S_0 \exp\left\{ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \int_{\tau=0}^{\tau=t} dz_\tau \right\} \tag{18}$$

And we arrive at the well-known result (where $\overset{d}{=}$ denotes equal in distribution, and $\tilde{\varepsilon} \sim N(0,1)$)

$$S_t \overset{d}{=} S_0 \exp\left\{ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t} \tilde{\varepsilon} \right\} \tag{19}$$

or

$$\ln(S_t) \overset{d}{=} \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\tilde{\varepsilon} \tag{20}$$

Therefore, the underlying asset is lognormally distributed with $S_t \sim \Lambda\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$. Thus,

$$E(S_t) = \exp\left\{\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \frac{\sigma^2 t}{2}\right\} = S_0 \exp\{\mu t\} \tag{21}$$

$$E(S_t^2) = E(S_t)^2 \exp\{\sigma^2 t\} = S_0^2 \exp\{(2\mu + \sigma^2)t\} \tag{22}$$

Substituting equation (22) into equation (1), we have the following results.

**Theorem 6. Expected Terminal Option Values: GBM with Geometric Drift**
Assuming geometric Brownian motion with geometric drift,

$$E_0[C_t] = S_0 \exp\{\mu t\}N(d_1) - XN(d_1 - B_1) \tag{23a}$$

$$E_0[P_t] = XN(-d_1 + B_1) - S_0 \exp\{\mu t\}N(-d_1) \tag{23b}$$

$$d_1 = \frac{\ln[S_0 \exp\{\mu t\}/X]}{B_1} + \frac{B_1}{2} = \frac{\ln[S_0/X] + \left(\mu + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{23c}$$

$$B_1 = \ln\left[E(S_t^2)/E(S_t)^2\right]^{1/2} = \sigma\sqrt{t} \tag{23d}$$

*Geometric Brownian Motion with Mean Reversion*
A mean reverting model is now illustrated where

$$S_t = S_0 + \int_0^t \alpha(L_\tau - S_\tau)d\tau + \int_0^t \sigma S_\tau dz_\tau \tag{24a}$$

and

$$L_t = L_0 + \int_0^t \mu L_\tau d\tau \tag{24b}$$

or

$$dS_t = \alpha(L_t - S_t)dt + \sigma S_t dz_t \tag{24c}$$

and

$$dL_t = \mu L_t dt \tag{24d}$$

Note that

$$L_t = L_0 \exp\{\mu t\} \tag{25}$$

Comparing this model with equation (3), we have $\mu_1(\tau) = -\alpha; \mu_2(\tau) = \alpha L_0 \exp\{\mu\tau\}; \sigma_1(\tau) = \sigma; \sigma_2(\tau) = 0$. Therefore, from equation (3c),

$$y_t = \exp\left\{\int_0^t \left(-\alpha - \frac{\sigma^2}{2}\right) d\tau + \sigma \int_0^t dz_\tau\right\} \tag{26}$$

and from equation (3d)

$$S_t = y_t \left(S_0 + \int_0^t \frac{\alpha L_0 \exp\{\mu\tau\}}{y_\tau} d\tau\right) \tag{27}$$

Substituting for $y_t$, noting $Z_t = \int_0^t dz_\tau$, and rearranging, we have

$$\begin{aligned} S_t &= S_0 \exp\left\{-\left(\alpha + \frac{\sigma^2}{2}\right)t + \sigma Z_t\right\} \\ &+ \alpha L_0 \exp\left\{-\left(\alpha + \frac{\sigma^2}{2}\right)t + \sigma Z_t\right\} \int_0^t \exp\left\{\left(\mu + \alpha + \frac{\sigma^2}{2}\right)\tau - \sigma Z_\tau\right\} d\tau \end{aligned} \tag{28}$$

Based on equations (3d) and Theorems 1 and 4 and a bit of algebra, we have the results presented in the following results.

**Theorem 7. Expected Terminal Option Values: GBM with Mean Reversion** Assuming geometric Brownian motion with mean reversion,

$$E_0[C_t] = E_0[S_t]N(d_1) - XN(d_1 - B_1) \tag{29a}$$

$$E_0[P_t] = XN(-d_1 + B_1) - S_0 \exp\{\mu t\}N(-d_1) \tag{29b}$$

$$d_1 = \frac{\ln[E(S_t)/X]}{B_1} + \frac{B_1}{2} \tag{29c}$$

$$B_1 = \ln\left[E\left(S_t^2\right)/E(S_t)^2\right]^{1/2} \tag{29d}$$

where

$$E(S_t) = S_0 \exp\{-\alpha t\} + \frac{\alpha}{\mu+\alpha} L_0 [\exp\{\mu t\} - \exp\{-\alpha t\}] \tag{29e}$$

The second moment requires more effort, we have[6]

$$\begin{aligned} E(S_t^2) &= S_0^2 \exp\{(\sigma^2 - 2\alpha)t\} + \frac{2\alpha S_0 L_0}{\mu+\alpha-\sigma^2}[\exp\{(\mu-\alpha)t\} - \exp\{(\sigma^2-2\alpha)t\}] \\ &+ 2\alpha^2 L_0^2 \left[ \frac{\exp\{(\sigma^2-2\alpha)t\}}{(\mu+\alpha-\sigma^2)(2\mu+2\alpha-\sigma^2)} + \frac{\exp\{2\mu t\}}{(\mu+\alpha)(2\mu+2\alpha-\sigma^2)} - \frac{\exp\{(\mu-\alpha)t\}}{(\mu+\alpha)(\mu+\alpha-\sigma^2)} \right] \end{aligned} \tag{29f}$$

## 4.2 Normal Distribution Applications

The results given in Theorem 2 and 5 are illustrated with two specific applications of linear normal models, arithmetic Brownian motion with arithmetic drift and arithmetic Brownian motion with geometric drift.

*Arithmetic Brownian Motion with Arithmetic Drift*
Consider an underlying asset where $\mu_1(\tau) = 0; \mu_2(\tau) = \mu; \sigma_1(\tau) = 0; \sigma_2(\tau) = \sigma$. Then based on Theorem 3 and equation (11a),

$$S_t = S_0 + \mu t + \sigma \int_{\tau=0}^{\tau=t} dz_\tau \tag{30a}$$

or

$$dS_t = \mu dt + \sigma dz_t \tag{30b}$$

This stochastic process is arithmetic Brownian motion (ABM) with arithmetic drift. Therefore, the terminal asset value distribution can be expressed as

$$S_t \stackrel{d}{=} S_0 + \mu t + \sigma\sqrt{t}\tilde{\varepsilon} \tag{31}$$

where $\tilde{\varepsilon} \sim N(0,1)$. Therefore, the underlying asset is normally distributed with $S_t \sim N(S_0 + \mu t, \sigma^2 t)$ and from Theorem 5, we have

$$E(S_t) = S_0 + \mu t \tag{32a}$$

$$E(S_t^2) = \sigma^2 t + (S_0 + \mu t)^2 \tag{32b}$$

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[6] Proof available upon request.

Substituting the first two moments into Theorem 2, we have the appropriate terminal expected option value for ABM with arithmetic drift.

**Theorem 8. Expected Terminal Option Values: ABM with Arithmetic Drift**
Assuming arithmetic Brownian motion with arithmetic drift,

$$E_0[C_t] = [S_0 + \mu t - X]N(d_n) + \sigma\sqrt{t}n(d_n) \tag{33a}$$

$$E_0[P_t] = [X - S_0 + \mu t]N(-d_n) + \sigma\sqrt{t}n(-d_n) \tag{33b}$$

$$d_n = \frac{S_0 + \mu t - X}{\sigma\sqrt{t}} \tag{33c}$$

*Arithmetic Brownian Motion with Geometric Drift*
Consider $\mu_1(\tau) = \mu; \mu_2(\tau) = 0; \sigma_1(\tau) = 0; \sigma_2(\tau) = \sigma$, then based on Theorem 3 and equation (11),

$$S_t = S_0 + \int_{\tau=0}^{\tau=t} \mu S_\tau d\tau + \sigma \int_{\tau=0}^{\tau=t} dz_\tau \tag{34a}$$

or

$$dS_t = \mu S_t dt + \sigma dz_t \tag{34b}$$

which is ABM with geometric drift. From Theorem 5, we have

$$E(S_t) = S_0 \exp\{\mu t\} \tag{35a}$$

$$E(S_t^2) = S_0^2 \exp\{2\mu t\} + \sigma^2 \frac{\exp\{2\mu t\} - 1}{2\mu} \tag{35b}$$

Substituting the first two moments we have the terminal expected option value.

**Theorem 9. Expected Terminal Option Values: ABM with Geometric Drift**
Assuming arithmetic Brownian motion with geometric drift,

$$E_0[C_t] = [S_0 \exp\{\mu t\} - X]N(d_n) + \sigma\sqrt{\frac{\exp\{2\mu t\} - 1}{2\mu}}n(d_n) \tag{36a}$$

$$E_0[P_t] = [X - S_0 \exp\{\mu t\}]N(-d_n) + \sigma\sqrt{\frac{\exp\{2\mu t\} - 1}{2\mu}}n(-d_n) \tag{36b}$$

$$d_n = \frac{S_0 \exp\{\mu t\} - X}{\sigma\sqrt{\frac{\exp\{2\mu t\} - 1}{2\mu}}} \tag{36c}$$

## 4 Conclusion

In this paper we have presented a methodology for deriving closed-form solutions for two categories of linear single factor terminal expected option values. Terminal expected option values are useful for a variety of option problems. Two linear single factor model applications based on either the lognormal distribution or the normal distribution are illustrated. These linear factor models are based on the reference instrument's stochastic differential equation is linear in the instrument, although not necessarily linear in time. For many option-related trading decisions, the analysis hinges critically on estimating the first two moments of the underlying distribution.

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