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A Note on Grüss Type Inequality

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Abstract. In this short note, we establish a new form of the inequality of Grüss type for functions whose first and second derivatives are absolutely continuous and third derivative is bounded both above and below almost everywhere.

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1. Introduction

Let f and g be two bounded functions defined on $[a, b]$ with $\gamma_1 \le f(x) \le \Gamma_1$ and $\gamma_2 \le g(x) \le \Gamma_2$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are four constants. Then the classic Grüss inequality reads as follows:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx \le \frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).$$

In the years thereafter, numerous generalizations, extensions and variants of Grüss inequality have appeared in the literature (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). The purpose of the present note is to establish a new form of the inequality of Grüss type for functions whose first and second derivatives are absolutely continuous and third derivative is bounded both above and below almost everywhere.

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2. Grüss inequality

In this section, we shall obtain the following main result.

Theorem 2.1. *Let $f : [a,b] \to (-\infty,\infty)$ be a function such that the derivative $f^{'}, f''$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in (-\infty,\infty)$ such that $\gamma \le f^{'''}(x) \le \Gamma$ a.e. on $[a,b]$. Then we have*

$$\begin{aligned}\Big|(a^2+ba+b^2)(bf^{"}(a)-af^{"}(b)) - 3(b^2f^{'}(b)-a^2f^{'}(a))\\ +6(bf(b)-af(a)) - \int_a^b f(x)dx\Big|\\ \le(\Gamma-\gamma)\frac{b^4+3C^{4/3}-4bC}{4}.\end{aligned}$$

where

$$C=\frac{(b+a)(b^2+a^2)}{4}.$$

Proof. Firstly, it is easy to check

$$\begin{aligned}&(a^2+ba+b^2)(bf^{"}(a)-af^{"}(b)) - 3(b^2f^{'}(b)-a^2f^{'}(a))\\ &\qquad+6(bf(b)-af(a)) - \int_a^b f(x)dx\\ &=b^3f^{''}(b)-a^3f^{''}(a)-3(b^2f^{'}(b)-a^2f^{'}(a))+6(bf(b)-af(a))\\ &\qquad-(b+a)(b^2+a^2)[f^{"}(b)-f^{"}(a)]-\int_a^b f(x)dx\\ &=\int_a^b\left\{x^3-\frac{1}{b-a}\int_a^b x^3dx\right\}f^{'''}(x)dx.\end{aligned}$$

Let

$$A=\left\{x\in[a,b]: x^3\ge\frac{1}{b-a}\int_a^b x^3dx\right\};$$

$$A^c=\left\{x\in[a,b]: x^3<\frac{1}{b-a}\int_a^b x^3dx\right\}.$$

Then we have

$$\begin{aligned}&\int_a^b\left\{x^3-\frac{1}{b-a}\int_a^b x^3dx\right\}f^{'''}(x)dx\\ \le&\Gamma\int_A\left\{x^3-\frac{1}{b-a}\int_a^b x^3dx\right\}dx+\gamma\int_{A^c}\left\{x^3-\frac{1}{b-a}\int_a^b x^3dx\right\}dx\end{aligned}$$

and

$$\int_a^b \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} f'''(x) dx$$

$$\geq \gamma \int_A \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} dx + \Gamma \int_{A^c} \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} dx.$$

Since

$$\int_A \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} dx = - \int_{A^c} \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} dx,$$

it follows that

$$\left| \int_a^b \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} f'''(x) dx \right| \tag{2.1}$$

$$\leq (\Gamma - \gamma) \int_A \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} dx$$

$$= (\gamma - \Gamma) \int_{A^c} \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} dx.$$

Therefore, it is enough to discuss the following integral,

$$\int_A \left\{ x^3 - \frac{1}{b-a} \int_a^b x^3 dx \right\} dx. \tag{2.2}$$

From the definition of the set A, it follows that

$$A = \left\{ x \in [a,b]; \sqrt[3]{\frac{(b+a)(b^2+a^2)}{4}} \leq x \leq b \right\},$$

and we can claim that

$$a \leq \sqrt[3]{\frac{(b+a)(b^2+a^2)}{4}} \leq b, \quad \forall\ a < b. \tag{2.3}$$

In fact, we can assume $b = ka$, where k is chosen from R based on a. If $a \geq 0$ which implies $b > 0$, then $k > 1$ and the inequality (2.3) is equivalent to

$$1 \leq \frac{(k+1)(k^2+1)}{4} \leq k^3$$

which is obvious. Similarly if $a < 0, b \leq 0$, then $0 \leq k \leq 1$ and the inequality (2.3) is equivalent to

$$1 \geq \sqrt[3]{\frac{(k+1)(k^2+1)}{4}} \geq k, \tag{2.4}$$

if $a < 0, b \geq 0$, then $k \leq 0$ and the inequality (2.3) is equivalent also to

$$1 \geq \sqrt[3]{\frac{(k+1)(k^2+1)}{4}} \geq k, \tag{2.5}$$

It is easy to see (2.4) and (2.5) hold correspondingly. Hence the integral (2.2) can be obtained,

$$\begin{aligned}(2.6)\qquad &\int_A \left\{x^3 - \frac{1}{b-a}\int_a^b x^3dx\right\}dx\\ &= \int_{\sqrt[3]{\frac{(b+a)(b^2+a^2)}{4}}}^{b} \left\{x^3 - \frac{1}{b-a}\int_a^b x^3dx\right\}dx\\ &= \frac{b^4 - \left(\frac{(b+a)(b^2+a^2)}{4}\right)^{4/3}}{4} - \frac{(a+b)(a^2+b^2)}{4}\left[b - \left(\frac{(b+a)(b^2+a^2)}{4}\right)^{1/3}\right]\\ &= \frac{b^4 + 3\left(\frac{(b+a)(b^2+a^2)}{4}\right)^{4/3} - 4b\frac{(b+a)(b^2+a^2)}{4}}{4}.\end{aligned}$$

The desired result can be obtained. □

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