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A Class of Parallel Iterative Method for 1D Diffusion Equation

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Abstract

In this paper, we first derive an absolutely stable implicit finite difference scheme with four order accurate in spatial step size and two order in time step size for diffusion equations. Based on the scheme we present a class of alternating group explicit iterative method. The method is suitable for parallel computation. Results of the convergence analysis shows that the method is convergent. In the end, several numerical examples are presented to confirm the analysis for the method.

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Keywords: diffusion equations, periodic boundary value, iterative method, parallel computation

1 Introduction

We consider the following periodic boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2},\ 0 \le t \le T \\ u(x,0) = f(x), \\ u(x,t) = u(x+1,t). \end{cases} \tag{1.1}$$

Many researches on numerical methods of diffusion equations have been done, but researches on finite difference methods for the periodic boundary value problem has scarcely presented. As we all know, Most of explicit methods are short in stability and accuracy, while implicit methods are unadaptable for parallel computing, and need to solve large equation set. Thus it is a great task to present parallel numerical methods with absolute stability. A class of alternating group method (AGE) for paraboic equations are presented in [1].

Because of the parallelism and absolute stability, the AGE method is widely cared and developed by many authors such as B.l.Zhang, G.W.Yuan etc in [2-4], while R. Tavakoli derived a class of domain-split method based on the AGE method for 1D and 2D diffusion equation in [5,6]. Most of the developed methods have the same advantage of good stability and parallelism, but almost all the methods have $O(h^2)$ accuracy for spatial step in the case of using six grid points.

We organize this paper as follows: In section 2, we present a $O(\tau^2+h^4)$ order unconditionally stable symmetry six-point implicit scheme for solving (1.1) at first. Based on the scheme two alternating group explicit iterative methods are constructed. In section 3, convergence analysis and stability analysis are given. In section 4, results of several numerical examples are presented.

2 The Alternating Group Iterative Method(AGI)

The domain $\Omega:[0,1]\times[0,T]$ will be divided into $(m\times\xi)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{\xi}$. Grid points are denoted by $(x_i,t_n),\ x_i=ih(i=0,\ 1,\cdots,m),\ t_n=n\tau(n=0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (1.1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$

Let $\delta_t u_j^n=\frac{u_j^{n+1}-u_j^n}{\tau},\ \delta_x^2 u_j^n=\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2}$. We present an implicit finite difference scheme with parameters for solving (1.1) as below:

$$\kappa_1\delta_t u_{j-1}^n+\kappa_2\delta_t u_j^n+\kappa_3\delta_t u_{j+1}^n=\frac{a}{2}(\delta_x^2 u_j^{n+1}+\delta_x^2 u_j^n) \tag{2.1}$$

Applying Taylor formula to the scheme at (x_i,t_n). Considering $\frac{\partial^k u}{\partial t^k}=a^k\frac{\partial^{2k}u}{\partial t^{2k}}$, then we have the truncation error:

$$(\kappa_1+\kappa_2+\kappa_3-1)a(\frac{\partial^2 u}{\partial x^2})_i^n+(-\kappa_1+\kappa_3)ah(\frac{\partial^3 u}{\partial x^3})_i^n-\frac{1}{2}(\kappa_1+\kappa_2+\kappa_3-1)a^2\tau(\frac{\partial^4 u}{\partial x^4})_i^n+$$

$$\frac{1}{2}(\kappa_1+\kappa_3-\frac{1}{6})ah^2(\frac{\partial^4 u}{\partial x^4})_i^n+\frac{1}{6}(\kappa_3-\kappa_1)ah^3(\frac{\partial^5 u}{\partial x^5})_i^n+\frac{1}{2}(\kappa_1-\kappa_3)a^2\tau h(\frac{\partial^5 u}{\partial x^5})_i^n+$$

$$\frac{1}{4}(-\kappa_1-\kappa_3+\frac{1}{6})a^2\tau h^2(\frac{\partial^6 u}{\partial x^6})_i^n+\frac{1}{12}(\kappa_1+\kappa_3-\frac{1}{6})a^2\tau h^3(\frac{\partial^7 u}{\partial x^7})_i^n+O(\tau^2+h^4)$$

Let

$$\begin{cases}\kappa_1+\kappa_2+\kappa_3-1=0\\ -\kappa_1+\kappa_3=0\\ \kappa_1+\kappa_3-\frac{1}{6}=0\end{cases}$$

that is, $\kappa_1=\kappa_3=\frac{1}{12},\ \kappa_2=\frac{5}{6}$. Then we can easily have that the truncation error of the scheme is $O(\tau^2+h^4)$.

Let $r=\frac{a\tau}{24h^2}$, then from (2.1) we have:

$$(1-6r)u_{i-1}^{n+1}+(10+12r)u_i^{n+1}+(1-6r)u_{i+1}^{n+1} = (1+6r)u_{i-1}^{n}+(10-12r)u_i^{n}+(1+6r)u_{i+1}^{n} \tag{2.2}$$

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$, then from (2.1) we can have $AU^{n+1} = FU^n = \widetilde{F}^n$. here

$$A = \begin{pmatrix} 10+12r & 1-6r & & & 1-6r \\ 1-6r & 10+12r & 1-6r & & \\ & \cdots & \cdots & \cdots & \\ & & 1-6r & 10+12r & 1-6r \\ 1-6r & & & 1-6r & 10+12r \end{pmatrix}_{m\times m}$$

$$F = \begin{pmatrix} 10-12r & 1+6r & & & 1+6r \\ 1+6r & 10-12r & 1+6r & & \\ & \cdots & \cdots & \cdots & \\ & & 1+6r & 10-12r & 1+6r \\ 1+6r & & & 1+6r & 10-12r \end{pmatrix}_{m\times m}$$

The alternating group iterative method will be constructed in two cases as follows:

First we let $m = 4k,\ k$ is an integer. Let $A = \frac{1}{2}(G_1 + G_2)$, here

$$G_1 = \begin{pmatrix} B_1 & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & B_1 \end{pmatrix}_{m\times m}, \quad G_2 = \begin{pmatrix} B_2 & & & & C_1 \\ & B_1 & & & \\ & & \cdots & & \\ & & & B_1 & \\ C_1^T & & & & B_2 \end{pmatrix}_{m\times m}$$

$$B_1 = \begin{pmatrix} 10+12r & 1-6r & 0 & 0 \\ 0 & 10+12r & 2(1-6r) & 0 \\ 0 & 2(1-6r) & 10+12r & 1-6r \\ 0 & 0 & 1-6r & 10+12r \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 10+12r & 1-6r \\ 1-6r & 10+12r \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 2(1-6r) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the alternating group iterative method I can be denoted as below:

$$\begin{cases} (\rho I + G_1)U^{n+1(k+\frac{1}{2})} = (\rho I - G_2)U^{n+1(k)} + 2\widetilde{F}^n \\ (\rho I + G_2)U^{n+1(k+1)} = (\rho I - G_1)U^{n+1(k+\frac{1}{2})} + 2\widetilde{F}^n \end{cases} \tag{2.3}$$

Here ρ is the iterative parameter. k is the iterative number, $k = 0, 1, \cdots$.

Second we let $m = 4k+2,\ k$ is an integer. Let $A = \frac{1}{2}(\overline{G}_1 + \overline{G}_2)$, here

$$\overline{G}_1 = \begin{pmatrix} B_1 & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & & & B_1 & \\ & & & & B_2 \end{pmatrix}_{m\times m}, \ \overline{G}_2 = \begin{pmatrix} B_2 & & & & C_2 \\ & B_1 & & & \\ & & \cdots & & \\ & & & \cdots & \\ C_2^T & & & & B_1 \end{pmatrix}_{m\times m}$$

$$B_1 = \begin{pmatrix} 10+12r & 1-6r & 0 & 0 \\ 0 & 10+12r & 2(1-6r) & 0 \\ 0 & 2(1-6r) & 10+12r & 1-6r \\ 0 & 0 & 1-6r & 10+12r \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 10+12r & 1-6r \\ 1-6r & 10+12r \end{pmatrix}, \ C_2 = \begin{pmatrix} 0 & 0 & 0 & 2(1-6r) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the alternating group iterative method II can be derived as below:

$$\begin{cases} (\rho I + \overline{G}_1)U^{n+1(k+\frac{1}{2})} = (\rho I - \overline{G}_2)U^{n+1(k)} + 2\widetilde{F}^n \\ (\rho I + \overline{G}_2)U^{n+1(k+1)} = (\rho I - \overline{G}_1)U^{n+1(k+\frac{1}{2})} + 2\widetilde{F}^n \end{cases}. \tag{2.4}$$

3 Convergence and Stability Analysis

Lemma[7] Let $\theta > 0$, and $G + G^T$ is positive, then $(\theta I + G)^{-1}$exists, and

$$\begin{cases} \|(\theta I + G)^{-1}\|_2 \le \theta^{-1} \\ \|(\theta I - G)(\theta I + G)^{-1}\|_2 < 1 \end{cases} \tag{3.1}$$

From the construction of the matrixes we can see that $G_1,\ G_2,\ (G_1 + G_1^T),\ (G_2 + G_2^T)$ are all nonnegative matrixes. Then we have

$$\|(\rho I - G_1)(\rho I + G_1)^{-1}\|_2 \le 1,\ \|(\rho I - G_2)(\rho I + G_2)^{-1}\|_2 \le 1$$

From (2.2), we can obtain $U^{n+1} = GU^n + 2(\rho I + G_2)^{-1}[(\rho I - G_1)(\rho I + G_1)^{-1}\widetilde{F}^n + \widetilde{F}^n]$. here $G = (\rho I + G_2)^{-1}(\rho I - G_1)(\rho I + G_1)^{-1}(\rho I - G_2)$ is the growth matrix.

Let $\widetilde{G} = (\rho I + G_2)G(\rho I + G_2)^{-1} = (\rho I - G_1)(\rho I + G_1)^{-1}(\rho I - G_2)(\rho I + G_2)^{-1}$, then $\rho(G) = \rho(\widetilde{G}) \le \|\widetilde{G}\|_2 \le 1.$, which shows the AGI1 method given by (2.3) is convergent. Then we have:

Theorem The alternating group iterative method I is convergent.

Analogously we have:

Theorem The alternating group iterative method II is convergent.

4 Numerical Experiments

We consider the following initial boundary value problem of diffusion equations:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 \le x \le 1, \ 0 \le t \le T \\ u(x,0) = sin(2\pi x), \\ u(0,t) = 0, \ u(1,t) = 0. \end{cases} \tag{4.1}$$

The exact solution for the problem is $u(x,t) = e^{-4\pi^2 t} sin(2\pi x)$. Let A.E denote maximum absolute error, while P.E denote maximum relevant error. A.E=$|u_i^n - u(x_i,t_n)|$, P.E=$100 \times |u_i^n - u(x_i,t_n)/u(x_i,t_n)|$. Let $\rho = 1$. In order to verify the alternating group method, we finished the following examples in variant conditions. We will use the iterative error 1×10^{-8} to control the process of iterativeness, and the results of numerical experiments are listed in the following tables:

Table 1: The numerical results for the iterative method I $m = 16$

	$\tau = 10^{-3}, t = 100\tau$	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 1000\tau$
A.E	5.477×10^{-5}	2.056×10^{-4}	5.392×10^{-4}
P.E	2.748×10^{-1}	3.144×10^{-2}	2.812×10^{-1}
average iterative times	23.53	21.52	16.344

Table 2: The numerical results for the iterative method I $m = 20$

	$\tau = 10^{-3}, t = 100\tau$	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 1000\tau$
A.E	4.120×10^{-5}	1.834×10^{-4}	5.185×10^{-4}
P.E	2.143×10^{-1}	2.730×10^{-2}	2.689×10^{-1}
average iterative times	23.77	21.79	16.573

Table 3: The numerical results for the iterative method II at $m = 18$

	$\tau = 10^{-3}, t = 100\tau$	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 1000\tau$
A.E	4.650×10^{-5}	1.877×10^{-4}	5.165×10^{-4}
P.E	2.447×10^{-1}	2.842×10^{-2}	2.718×10^{-1}
average iterative times	23.71	21.72	16.507

Table 4: The numerical results for the iterative method II at $m = 22$

	$\tau = 10^{-3}, t = 100\tau$	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 1000\tau$
A.E	3.433×10^{-5}	1.803×10^{-4}	5.191×10^{-4}
P.E	1.802×10^{-1}	2.707×10^{-2}	2.719×10^{-1}
average iterative times	23.74	21.79	16.561

The results of table 1-4 shows that the alternating group methods are converge, and don't lead to numerical instability in any cases in the process of computation, which accords to the conclusion of convergence and error analysis. Besides those, from the construction of the AGI methods we notice that the AGI methods are suitable for parallel computation obviously.

5 Conclusions

In this paper, we present two alternating group iterative(AGI) methods for diffusion equations. The methods are based on an $O(\tau^2 + h^4)$ order implicit scheme, which is of absolute stability. Of course we can establish another alternating group iterative method based on other implicit schemes, which shows the construction of the AGI methods mentioned in this paper is a universal process, and the methods can also be applied to other partial differential equations. Considering the parallelism, the AGI methods are convenient to use in solving large equation set.

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