

Deteriorating Inventory Model for Waiting Time Partial Backlogging

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Abstract. In this study, a deterministic inventory model in which items are subject to constant deterioration and shortages are allowed. The unsatisfied demand is backlogged which is a function of time. The optimal order quantity is derived by minimizing the total cost. The numerical example is given to support the result. The convexity of the cost function is shown numerically. Sensitivity analysis is carried out to analyze the effect of critical parameters on decision variables and the total cost of an inventory system.

Mathematics Subject Classification: 90B05

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1. Introduction:

Deterioration is defined as decay, spoilage, loss of utility of the product. The process of deterioration is observed in volatile liquids, beverages, medicines, blood components, food stuffs, dairy items etc. As a result, while determining optimal replenishment order, the loss due to deterioration can not be avoided. The literature survey by Raafat (1991), Shah and Shah (2000) and Goyal and Giri (2001) cite upto date review on deteriorating inventory models. Another scenario of inventory model is about stock outs. In market, it is observed that because of good reputation of the retailer, some customers are willing to wait for new stocks arrival, or if the wait will be short, while other may go elsewhere. Abad (1996, 2001) derived a pricing and ordering policy for a variable rate of deterioration and partially backlogging. The partial backlogging was assumed to be exponential function of waiting time till the next replenishment. The assumptions of exponential backlogging are unrealistic in developing countries. Secondly, Abad's article does not include backorder cost and lost sale cost in the formulation of the

objective function which influences service level to customers. Dye et al (2007) took into account the backorder cost and lost sale.

The aim of the paper is to make the paper of Abad (1996, 2001) more realistic and applicable in practice. It is assumed that the backlogged units are proportional to the waiting time. The rest of the paper is organized as follows. In the next section, the assumptions and notations are listed for the aforesaid concept. In section 3, mathematical model is derived. Section 4 deals with numerical example and sensitivity analysis. The concluding remarks are given in section 5.

2. Assumptions and Notations:

The mathematical model is based on the following notations and assumptions.

2.1 Notations:

| | |
|---------------|--|
| A | the ordering cost per order |
| C | the purchase cost per unit. |
| h | the inventory holding cost per unit per time unit |
| π_b | the backordered cost per unit short per time unit. |
| π_L | the cost of lost sales per unit. |
| t_1 | the time at which the inventory level reaches zero, $t_1 \geq 0$ |
| t_2 | the length of period during which shortages are allowed, $t_2 \geq 0$ |
| T | (= $t_1 + t_2$) the length of cycle time |
| IM | the maximum inventory level during $[0, T]$. |
| IB | the maximum backordered units during stock out period. |
| Q | (= $IM + IB$) the order quantity during a cycle of length T. |
| $I_1(t)$ | the level of positive inventory at time t, $0 \leq t \leq t_1$ |
| $I_2(t)$ | the level of negative inventory at time t, $t_1 \leq t \leq t_1 + t_2$ |
| $K(t_1, t_2)$ | the total cost per time unit. |

2.2 Assumptions:

1. The inventory system deals with single item.
2. The demand rate 'a' is known and constant.
3. The replenishment rate is infinite.
4. The lead – time is zero or negligible.
5. The planning horizon is infinite.
6. During the stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for the next replenishment. (Ouyang et al (2005)). The proportion of the customers who would like

to accept the backlogging at time t is with the waiting time $(T - t)$ for the next replenishment, i.e. for the negative inventory the backlogging rate is $B(t) = \frac{1}{1 + \delta(T - t)}$; $\delta > 0$ denotes the backlogging parameter and $t_1 \leq t \leq T$

3. Mathematical Model:

Under above assumption, the on – hand inventory level at any instant of time is exhibited in figure 1.

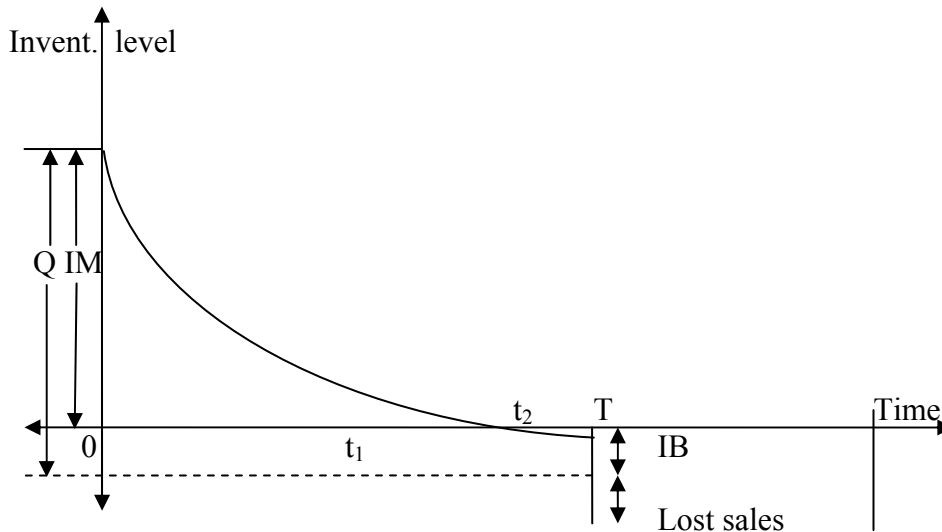


Figure 1 Representation of inventory system

During the period $[0, t_1]$, the inventory depletes due to the cumulative effects of demand and deterioration. Hence, the inventory level at any instant of time during $[0, t_1]$ is described by the differential equation

$$\frac{dI_1(t)}{dt} = -a - \theta I_1(t) \quad ; \quad 0 \leq t \leq t_1 \tag{3.1}$$

with the boundary condition $I_1(t_1) = 0$, the solution of differential equation (3.1) is

$$I_1(t) = \frac{a}{\theta} (e^{\theta(t_1-t)} - 1) \quad ; \quad 0 \leq t \leq t_1 \tag{3.2}$$

From time t_1 onwards shortages occur and inventory level reaches to zero. During the interval $[t_1, t_1 + t_2]$, the inventory level depends on demand and a fraction of

the demand is backlogged. The state of inventory during $[t_1, t_1 + t_2]$ can be represented by the differential equation,

$$\frac{dI_2(t)}{dt} = \frac{-a}{1 + \delta(t_1 + t_2 - t)}; \quad t_1 \leq t \leq t_1 + t_2 \quad (3.3)$$

Using $I_2(t_1) = 0$, the solution of differential equation is

$$I_2(t) = \frac{a}{\delta} (\ln(1 + \delta(t_1 + t_2 - t)) - \ln(1 + \delta t_2)) \quad (3.4)$$

The maximum positive inventory is

$$IM = I_1(0) = \frac{a}{\theta} (e^{\theta t_1} - 1) \quad (3.5)$$

The maximum backordered units are

$$IB = -I_2(t_1 + t_2) = \frac{a}{\delta} \ln(1 + \delta t_2) \quad (3.6)$$

Hence, the order size during $[0, T]$ is $Q = IM + IB$.

$$Q = \frac{a}{\theta} (e^{\theta t_1} - 1) + \frac{a}{\delta} \ln(1 + \delta t_2) \quad (3.7)$$

The total cost per cycle consists of following cost components.

1. Ordering cost per cycle; $OC = A$
2. Inventory holding cost per cycle;

$$\begin{aligned} IHC &= h \int_0^{t_1} I_1(t) dt \\ &= \frac{ha}{\theta^2} (e^{\theta t_1} - 1 - \theta t_1) \end{aligned}$$

3. Backordered cost per cycle;

$$\begin{aligned} BC &= \pi_b \int_{t_1}^{t_1+t_2} -I_2(t) dt \\ &= \pi_b \frac{a}{\delta^2} (\delta t_2 - \ln(1 + \delta t_2)) \end{aligned}$$

4. cost due to lost sales per cycle;

$$\begin{aligned} LS &= \pi_L a \int_{t_1}^{t_1+t_2} \left(1 - \frac{1}{1 + \delta(t_1 + t_2 - t)}\right) dt \\ &= \pi_L \frac{a}{\delta} (\delta t_2 - \ln(1 + \delta t_2)) \end{aligned}$$

5. Purchase cost per cycle;

$$\begin{aligned}
 PC &= C \times Q \\
 &= C\left(\frac{a}{\theta}(e^{\theta t_1} - 1) + \frac{a}{\delta} \ln(1 + \delta t_2)\right)
 \end{aligned}$$

Therefore, the total cost per time unit is

$$K(t_1, t_2) = \frac{1}{(t_1 + t_2)} [OC + IHC + BC + LS + PC] \tag{3.8}$$

The necessary condition for the total cost per time unit, to be minimize is

$$\begin{aligned}
 \frac{\partial K}{\partial t_1} &= \frac{ha(e^{\theta t_1} - 1) + cae^{\theta t_1}}{\theta(t_1 + t_2)} - \frac{A\theta^2 - ha(1 + \theta t_1 - e^{\theta t_1})}{\theta^2(t_1 + t_2)^2} \\
 &\quad - \frac{a(\delta t_2 - \ln(1 + \delta t_2))(\pi_b + \pi_L \delta)}{\delta^2(t_1 + t_2)^2} - \frac{ca(\delta(e^{\theta t_1} - 1) + \theta \ln(1 + \delta t_2))}{\delta \theta(t_1 + t_2)^2} = 0 \tag{3.9}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial K}{\partial t_2} &= \frac{\pi_b a t_2 + \pi_L \delta a t_2 + ca}{(t_1 + t_2)(1 + \delta t_2)} - \frac{A\theta^2 - ha(1 + \theta t_1 - e^{\theta t_1})}{\theta^2(t_1 + t_2)^2} \\
 &\quad - \frac{a(\delta t_2 - \ln(1 + \delta t_2))(\pi_b + \pi_L \delta)}{\delta^2(t_1 + t_2)^2} - \frac{ca(\delta(e^{\theta t_1} - 1) + \theta \ln(1 + \delta t_2))}{\delta \theta(t_1 + t_2)^2} = 0 \tag{3.10}
 \end{aligned}$$

Provided

$$\left(\frac{\partial^2 K}{\partial t_1^2}\right)\left(\frac{\partial^2 K}{\partial t_2^2}\right) - \left(\frac{\partial^2 K}{\partial t_1 \partial t_2}\right) > 0 \tag{3.11}$$

for obtained pair of (t_1, t_2) . The equations (3.9) and (3.10) are highly non – linear.

Using mathematical software, for given set of parametric equations (3.9) and (3.10) can be solved. The obtained values of t_1 and t_2 must satisfy equation (3.11) to minimize the total cost per time unit of an inventory system.

To illustrate and validate the proposed model, let us consider a numerical data in the following section and carry out sensitivity analysis with respect to backlogging parameter, deterioration rate and demand.

4. Numerical example and Sensitivity Analysis:

Consider an inventory system with following parametric values in proper units.

$$[A, C, h, \pi_b, \pi_L, a, \delta] = [250, 8, 0.50, 12, 15, 25, 2]$$

when deterioration rate is 5 %, time; t_1 at which positive inventory is zero is 4.28 units and stock out period; t_2 is of length 0.25 units. This advises retailer to buy 124 units which will minimum cost \$ 307.50. The three dimensional total cost per time unit graph is shown in Figure 2 by plotting t_1 in the range of [4.25, 4.30] and t_2 in the range of [0.20, 0.30]. The graph given in Figure 2 indicates that total cost per time unit is strictly convex.

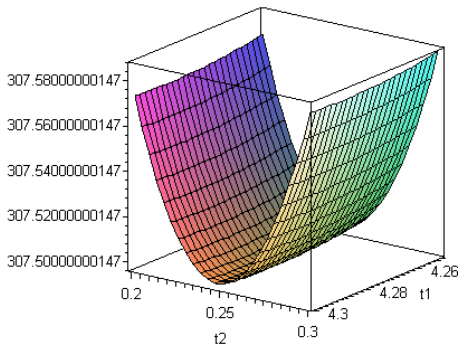


Figure 2 Total cost per time unit

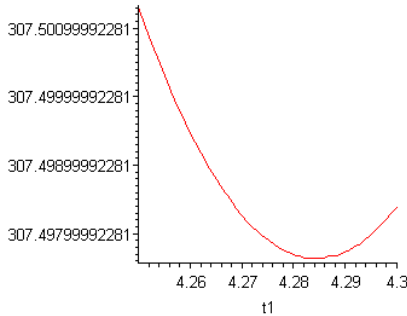


Figure 3 ($t_1 \rightarrow K$ with t_2 fixed)

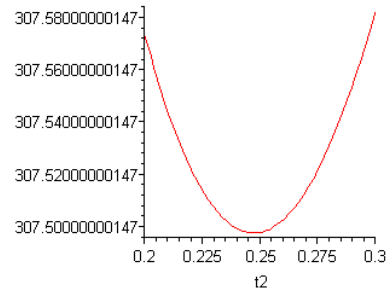


Figure 4 ($t_2 \rightarrow K$ with t_1 fixed)

Table 1 Variation in deterioration rate ‘ θ ’

| θ | t_1 | t_2 | Q | K |
|----------|-------|-------|-----|--------|
| 0.05 | 4.28 | 0.25 | 124 | 307.49 |
| 0.10 | 3.37 | 0.33 | 107 | 330.28 |
| 0.15 | 2.82 | 0.42 | 95 | 352.79 |

It is observed in table 1 that increase in deterioration rate increases shortages and total cost per time unit of an inventory system and decreases positive inventory time period and procurement quantity. See figures 3 and 4.

Table 2 Variation in backlogging parameter ' δ '

| δ | t_1 | t_2 | Q | K |
|----------|-------|-------|-----|--------|
| 5 | 4.32 | 0.17 | 124 | 308.45 |
| 10 | 4.34 | 0.11 | 124 | 309.09 |
| 15 | 4.36 | 0.06 | 123 | 309.65 |

Increase in backlogging parameter decreases positive inventory time period and total cost per time unit of an inventory system.

Table 3 Variation in demand ' a '

| α | t_1 | t_2 | Q | K |
|----------|-------|-------|-----|---------|
| 50 | 3.09 | 0.15 | 174 | 550.79 |
| 100 | 2.23 | 0.09 | 244 | 1011.94 |
| 150 | 1.59 | 0.06 | 343 | 1898.37 |

Increase in demand increases procurement quantity and total cost per time unit of an inventory system significantly.

5. Concluding Remarks:

In this study, an optimal replenishment schedule is derived under the assumption of waiting time backordering when units in an inventory are subject to constant deterioration. The model exhibits prevailing realistic market.

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