Applied Mathematical Sciences, Vol. 3, 2009, no. 4, 195 - 203

Alternating Direction Method for a Kind of Hyperbolic Integro-Differential Problem with Memory*

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Abstract

In this paper, a full-discrete alternating direction finite element method is presented for a class of hyperbolic integro-differential equation, the theocratical analysis show that the scheme is convergence, the optimal order error estimates in H^1 norm is obtained.

Keywords: hyperbolic integro-differential equation; alternating direction finite element method; error estimates

1 Introduction

We consider the following hyperbolic integro-differential problem with memory:

$$\begin{cases} u_{tt} - \nabla \cdot \{a(x,u)\nabla u + \int_0^t b(t,s,u(x,s))\nabla u ds\} = f(x,t,u) \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0. \end{cases} \tag{1}$$

where $x = (x_1, x_2), \Omega \in R^2$ is a bounded domain with smooth boundary $\partial\Omega$. We can assume $\Omega = [0,1] \times [0,1]$, refer to [1]. a, b, f, u_0, u_1 are prescribed. Let $J = [0,T]$, for each $(x,u) \in \Omega \times R$, we assume that the above problem satisfies the following conditions:
(i)there exist positive constants C_0, C_1,such that $0 < C_0 \le a(x,u) \le C_1$,and

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$a_{ij} = a_{ji}, i, j = 1, 2.$
(ii) a, b, f are all bounded together with their derivatives we needed.

(iii) $u, u_t \in L^\infty(J; H^{r+1} \cap W^{1,\infty}), u_{tt} \in L^2(J; H^{r+1}) \cap L^\infty(J; L^\infty), u_{ttt} \in L^2(J, H^1).$

Let $\tau_1^h = \{I_{1i}\}_{i=1}^{M_1}$, $\tau_2^h = \{I_{2j}\}_{j=1}^{M_2}$ denote quasi-uniform partitions of $I_1 = [0, 1]$and $I_2 = [0, 1]$, respectively, where $I_{1i} = [x_{1,i-1}, x_{1,i}], I_{2j} = [x_{2,j-1}, x_{2,j}]$.
Then $\bar{\Omega_h} = \tau_1^h \times \tau_2^h$ is a rectangular partition of $\bar{\Omega}$.
Assume $h_\iota(\iota = 1, 2)$ are the mesh parameters of $\tau_\iota^h, h = \max\{h_1, h_2\}$,where $0 < h \le h_0 \le 1$.We donote:

$$S_1^h = \{v|v \in H_0^1(I_1) \cap C(\bar{I}_1), v|_{I_{1i}} \in P_{r+1}(I_{1i}), \forall I_{1i} \in \tau_1^h\}$$

$$S_2^h = \{v|v \in H_0^1(I_2) \cap C(\bar{I}_2), v|_{I_{2j}} \in P_{r+1}(I_{2j}), \forall I_{2j} \in \tau_2^h\}$$

where r is a integer,$P_{r+1}(J)$ denote polynomials of degree at most $r + 1$.
Let $S_h = S_1^h \otimes S_2^h$.The following properties hold[2][3] :

$$\forall \phi \in H^{r+1} \cap H_0^1, \quad \inf_{v \in S_h} \{\|\phi - v\| + h\|\nabla(\phi - v)\|\} \le Ch^{r+1}\|\phi\|_{r+1},$$

$$\|v\|_{L^\infty} \le Ch^{-1}\|v\|, \quad \|\frac{\partial^2 v}{\partial x_1 \partial x_2}\| \le Ch^{-2}\|v\|, \quad \|v\|_1 \le Ch^{-1}\|v\|$$

Now we define the $Ritz - volterra$ projection operator. For any given function $u \in H_0^1(\Omega)$ and $t \in [0, T]$, define a function $\tilde{u} \in S_h$,such that:

$$(a(u)\nabla(u - \tilde{u}), \nabla v) + \left(\int_0^t b(t, s, u(x, s))\nabla(u - \tilde{u})ds, \nabla v\right) = 0, \quad \forall v \in S_h \quad (2)$$

Refer to [3], the $Ritz - volterra$ projection operator satisfies the properties:

$$\|u - \tilde{u}\| + \|(u - \tilde{u})_t\| + \|(u - \tilde{u})_{tt}\| + \|(u - \tilde{u})_{ttt}\| \le Ch^r \quad (3)$$

$$\|\nabla\tilde{u}\|_{0,\infty} + \|\nabla\tilde{u}_t\|_{0,\infty} + \|\nabla\tilde{u}_{tt}\|_{0,\infty} + \|\tilde{u}\|_{0,\infty} + \|\tilde{u}_t\|_{0,\infty} + \|\tilde{u}_{tt}\|_{0,\infty} \le C \quad (4)$$

Let Δt be the time step, $N = [T/\Delta t], t^n = n \cdot \Delta t, n = 0, 1, \cdots, N$.We denote:

$$U^n = U(t^n), \quad \Phi^n = \Phi(x, t^n), \quad f(\Phi^n) = f(x, t^n, \Phi^n)$$

$$d_t\Phi^n = \frac{\Phi^{n+1} - \Phi^n}{\Delta t}, \quad \partial^2\Phi^n = \frac{\Phi^{n+1} - 2\Phi^n + \Phi^{n-1}}{\Delta t^2}$$

We assume $\Delta t = O(h)$ in this paper.

2 Alternating Direction Garlerkin method

The weak form of (1) is to find a map $u(t) : [0, T] \rightarrow H_0^1(\Omega)$,such that

$$\begin{cases} (u_{tt}, v) + (a(x, u)\nabla u + \int_0^t b(t, s, u(x, s))\nabla u ds, \nabla v) = (f(x, t, u), v) \\ (u(x, 0), v) = (u_0, v), \quad (u_t(x, 0), v) = (u_1, v), \quad v \in H_0^1(\Omega), t \in [0, T], \\ u(\cdot, t) \in H_0^1(\Omega), \end{cases} \tag{5}$$

The Alternating Direction Garlerkin scheme can now be formulated as follows: find $U^{n+1} \in S_h, n = 1, \cdots, N-1$, such that

$$\begin{aligned} &(\frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2}, v) + \lambda(\nabla(U^{n+1} - 2U^n + U^{n-1}), \nabla v) \\ &+\lambda^2 \Delta t^2 (\frac{\partial^2}{\partial x_1 \partial x_2}(U^{n+1} - 2U^n + U^{n-1}), \frac{\partial^2 v}{\partial x_1 \partial x_2}) + (a(U^n)\nabla U^n, \nabla v) \\ &= (f(U^n), v) - (\Delta t \sum_{i=0}^{n} b(t_n, t_i, U^i)\nabla U^i, \nabla v) \end{aligned} \tag{6}$$

The initial approximation U^0 is the projection of $U_0(x)$, and U^1 satisfies the following equation(we denote $w = (\frac{\partial u}{\partial t})_0 \Delta t + \frac{1}{2}(\frac{\partial^2 u}{\partial t^2})_0 \Delta t^2$):

$$\begin{aligned} &(U^1 - U^0, v) + \Delta t(\nabla(U^1 - U^0), \nabla v) + \Delta t^4 (\frac{\partial^2}{\partial x_1 \partial x_2}(U^1 - U^0), \frac{\partial^2 v}{\partial x_1 \partial x_2}) \\ &= (w, v) + \Delta t(\nabla w, \nabla v) + \Delta t^4 (\frac{\partial^2 w}{\partial x_1 \partial x_2}, \frac{\partial^2 v}{\partial x_1 \partial x_2}) \end{aligned}$$

Assume $\{\alpha_\mu(x_1)\}_{\mu=1}^{M_1}$ forms a basis for S_1^h,$\{\beta_\kappa(x_2)\}_{\kappa=1}^{M_2}$ forms a basis for S_2^h, then we can assume $U = \sum\limits_\mu \sum\limits_\kappa \theta_{\mu\kappa}(t)\alpha_\mu(x_1)\beta_\kappa(x_2)$.let $v = \alpha_i(x_1)\beta_j(x_2)$, also

$$\zeta_{i\mu} = \int_a^b \alpha_i(x_1)\alpha_\mu(x_1)dx_1 \qquad \zeta'_{i\mu} = \int_a^b \frac{d\alpha_i}{dx_1}\frac{d\alpha_\mu}{dx_1}dx_1 \qquad \eta_{j\kappa} = \int_c^d \beta_j(x_2)\beta_\kappa(x_2)dx_2$$

$$\eta'_{j\kappa} = \int_c^d \frac{d\beta_j}{dx_2}\frac{d\beta_\kappa}{dx_2}dx_2 \qquad b^n = (b_1^n, b_2^n, \cdots, b_{M_1}^n) \qquad b_i^n = (b_{i1}^n, b_{i2}^n, \cdots, b_{iM_2}^n)$$

The (6) $\times$ $(\Delta t)^2$ can be written as follows:

$$\begin{aligned} &\sum_\mu \sum_\kappa (\zeta_{i\mu} + \lambda(\Delta t)^2 \zeta'_{i\mu})(\eta_{j\kappa} + \lambda(\Delta t)^2 \eta'_{j\kappa})\theta_{\mu\kappa}^{n+1} \\ &= \sum_\mu \sum_\kappa (\zeta_{i\mu} + \lambda(\Delta t)^2 \zeta'_{i\mu})(\eta_{j\kappa} + \lambda(\Delta t)^2 \eta'_{j\kappa})(2\theta_{\mu\kappa}^n - \theta_{\mu\kappa}^{n-1}) \\ &- \sum_\mu \sum_\kappa (a(\sum_\mu \sum_\kappa \theta_{\mu\kappa}^n)\nabla(\theta_{\mu\kappa}^n \alpha_\mu \beta_\kappa), \nabla(\alpha_i \beta_j))(\Delta t)^2 \\ &+(f(\sum_\mu \sum_\kappa \theta_{\mu\kappa}^n \alpha_\mu \beta_\kappa), \alpha_i \beta_j)(\Delta t)^2 - (\sum_\mu \sum_\kappa (\Delta t b^n \nabla(\theta_{\mu\kappa}\alpha_\mu \beta_\kappa)), \nabla(\alpha_i \beta_j))(\Delta t)^2 \end{aligned}$$

Let R_{ij}^ndenote the right-hand side of the above equation,we can get:

$$\sum_\mu(\zeta_{i\mu}+\lambda(\Delta t)^2\zeta'_{i\mu})Z_{\mu j}^{n+1}=R_{ij}^n,\qquad \sum_\kappa(\eta_{j\kappa}+\lambda(\Delta t)^2\eta'_{j\kappa})\theta_{\mu\kappa}^{n+1}=Z_{\mu j}^{n+1}$$

It is easy to know that the above two coefficient matrix are all symmetrical , positive definite. So,there exist unique solutions.

3 The error estimate

let $\eta^n=u^n-\tilde{u}^n,\xi^n=U^n-\tilde{u}^n$. It is easy to know that

$$\|\xi^0\|_1=0,\qquad \|\xi^1\|_1+\|d_t\xi^0\|\le C\Delta t \tag{7}$$

Combining (5) with (6), we can get the error equation:

$$\begin{aligned}
&(\frac{\xi^{n+1}-2\xi^n+\xi^{n-1}}{\Delta t^2},v)+\lambda(\nabla(\xi^{n+1}-2\xi^n+\xi^{n-1}),\nabla v)\\
&+\lambda^2\Delta t^2(\frac{\partial^2}{\partial x_1\partial x_2}(\xi^{n+1}-2\xi^n+\xi^{n-1}),\frac{\partial^2 v}{\partial x_1\partial x_2})\\
&=(\frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{\Delta t^2},v)+(f(U^n)-f(u^n),v)\\
&+(a(u^n)\nabla\tilde{u}^n-a(U^n)\nabla U^n,\nabla v)-\lambda(\nabla(\tilde{u}^{n+1}-2\tilde{u}^n+\tilde{u}^{n+1}),\nabla v)\\
&-\lambda^2\Delta t^2(\frac{\partial^2}{\partial x_1\partial x_2}(\tilde{u}^{n+1}-2\tilde{u}^n+\tilde{u}^{n-1}),\frac{\partial^2 v}{\partial x_1\partial x_2})+(u_{tt}^n-\frac{u^{n+1}-2u^n+u^{n-1}}{\Delta t^2},v)\\
&+(\int_0^{t_n}b(t_n,s,u(x,s))\nabla\tilde{u}^n ds-\Delta t\sum_{i=1}^n b(t_n,t_i,U^i)\nabla U^i,\nabla v)=\sum_{i=1}^7 I_i
\end{aligned}\tag{8}$$

Let $Q=\|\frac{\partial\tilde{u}}{\partial t}\|_{L^\infty}+1$, we can choose $\Delta t,h$, such that $C\Delta th^{-1}\le Q$.
Combining (7)with inverse properties we know:
$\|d_t\xi^0\|_{L^\infty}\le Ch^{-1}\|d_t\xi^0\|\le C\Delta th^{-1}\le Q$, then $\|d_tU^0\|_{L^\infty}\le 2Q$.
We assume: $\max\limits_{0\le n\le N-2}\|d_t\xi^n\|_{L^\infty}\le Q$ then, $\max\limits_{0\le n\le N-2}\|d_tU^n\|_{L^\infty}\le 2Q$.
We choose the test function $v=\xi^{n+1}-\xi^{n-1}=\Delta t(d_t\xi^{n-1}+d_t\xi^{n+1})$ to estimate the both side of (8).

$$I_1+I_6\le C\Delta t((\Delta t)^4+\|\partial^2\eta^n\|^2+\|d_t\xi^n\|^2+\|d_t\xi^{n-1}\|^2)$$

$$I_2\le C\Delta t(\|\xi^n\|^2+\|\eta^n\|^2+\|d_t\xi^n\|^2+\|d_t\xi^{n-1}\|^2)$$

$$\begin{aligned}
I_3&=((a(u^n)-a(U^n))\nabla\tilde{u}^n,\nabla(\xi^{n+1}-\xi^{n-1}))-(a(U^n)\nabla\xi^n,\nabla(\xi^{n+1}-\xi^{n-1}))\\
&=I_3^{(1)}+I_3^{(2)}
\end{aligned}\tag{9}$$

$$\sum_{n=1}^{N-1} I_3^{(1)} = \Delta t \sum_{n=1}^{N-1}((a(u^n) - a(U^n))\nabla \tilde{u}^n, \nabla d_t\xi^n)$$
$$+\Delta t \sum_{n=1}^{N-1}((a(u^n) - a(U^n))\nabla \tilde{u}^n, \nabla d_t\xi^{n-1}) = Q_1 + Q_2$$

$$Q_1 = ((a(u^{N-1}) - a(U^{N-1}))\nabla \tilde{u}^{N-1}, \nabla \xi^N) - ((a(u^1) - a(U^1))\nabla \tilde{u}^1, \nabla \xi^1)$$
$$-\Delta t \sum_{n=1}^{N-2}(\frac{a(u^{n+1}) - a(U^{n+1}) - a(u^n) + a(U^n)}{\Delta t}\nabla \tilde{u}^{n+1}$$
$$+\frac{\nabla \tilde{u}^{n+1} - \nabla \tilde{u}^n}{\Delta t}(a(u^n) - a(U^n)), \nabla \xi^{n+1})$$

$$((a(u^{N-1}) - a(U^{N-1}))\nabla \tilde{u}^{N-1}, \nabla \xi^N) - ((a(u^1) - a(U^1))\nabla \tilde{u}^1, \nabla \xi^1)$$
$$\leq C\Delta t \sum_{n=1}^{N-2}(\|d_t\xi^n\|^2 + \|d_t\eta^n\|^2) + \epsilon\|\xi^N\|_1^2 + C(\|\xi^1\|_1^2 + \|\eta^1\|^2)$$

Combining differential theorem of mean with (i),we can get

$$\Delta t \sum_{n=1}^{N-2}(\frac{a(u^{n+1}) - a(U^{n+1}) - a(u^n) + a(U^n)}{\Delta t}\nabla \tilde{u}^{n+1}$$
$$+\frac{\nabla \tilde{u}^{n+1} - \nabla \tilde{u}^n}{\Delta t}(a(u^n) - a(U^n)), \nabla \xi^{n+1})$$
$$\leq C\Delta t \sum_{n=1}^{N-2}(\|d_t\xi^n\|^2 + \|d_t\eta^n\|^2 + \|\xi^n\|^2 + \|\eta^n\|^2 + \|\xi^{n+1}\|_1^2 + \|\eta^{n+1}\|^2)$$

So,

$$Q_1 \leq \epsilon\|\xi^N\|_1^2 + C(\|\xi^1\|_1^2 + \|\eta^1\|^2) + C\Delta t \sum_{n=1}^{N-2}(\|d_t\xi^n\|^2 + \|d_t\eta^n\|^2 + \|\xi^n\|^2$$
$$+\|\eta^n\|^2 + \|\xi^{n+1}\|_1^2 + \|\eta^{n+1}\|^2) \tag{10}$$

We estimate Q_2 as the same as Q_1:

$$Q_2 \leq \epsilon\|\xi^{N-1}\|_1^2 + C(\|\xi^1\|^2 + \|\eta^1\|^2) + C\Delta t \sum_{n=1}^{N-2}(\|d_t\xi^n\|^2 + \|d_t\eta^n\|^2 + \|\xi^n\|_1^2$$
$$+\|\eta^n\|^2 + \|\xi^{n+1}\|^2 + \|\eta^{n+1}\|^2) \tag{11}$$

$$\sum_{n=1}^{N-1} I_3^{(1)} \leq \epsilon(\|\xi^N\|_1^2 + \|\xi^{N-1}\|_1^2) + C(\|\xi^1\|_1^2 + \|\eta^1\|^2 + \|\xi^1\|^2) + C\Delta t \sum_{n=1}^{N-2}(\|d_t\xi^n\|^2$$
$$+\|d_t\eta^n\|^2 + \|\xi^n\|_1^2 + \|\eta^n\|^2 + \|\xi^{n+1}\|^2 + \|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n+1}\|_1^2) \tag{12}$$

$$
\begin{aligned}
&I_4 = -\lambda(\nabla(\eta^{n+1} - 2\eta^n + \eta^{n-1}), \nabla(\xi^{n+1} - \xi^{n-1}))\\
&+\lambda(\nabla(u^{n+1} - 2u^n + u^{n-1}), \nabla(\xi^{n+1} - \xi^{n-1}))\\
&= -\lambda\Delta t^2(\nabla\partial^2\eta^n, \nabla(\xi^{n+1} - \xi^{n-1})) + \lambda(\nabla(u^{n+1} - 2u^n + u^{n-1}), \nabla(\xi^{n+1} - \xi^{n-1}))
\end{aligned}
$$

By inverse properties and integration by parts,we know

$$
I_4 \leq C\Delta t((\Delta t)^4 + \|d_t\xi^n\|^2 + \|d_t\xi^{n-1}\|^2 + (\Delta t)^4 h^{-2}\|\partial^2\eta^n\|_1^2) \tag{13}
$$

$$
\begin{aligned}
&I_5 = \lambda^2(\Delta t)^4(\frac{\partial^2}{\partial x_1\partial x_2}\partial^2 u^n, \Delta t\frac{\partial^2}{\partial x_1\partial x_2}(d_t\xi^n + d_t\xi^{n-1}))\\
&-\lambda^2(\Delta t)^4(\frac{\partial^2}{\partial x_1\partial x_2}\partial^2\eta^n, \Delta t\frac{\partial^2}{\partial x_1\partial x_2}(d_t\xi^n + d_t\xi^{n-1}))\\
&\leq C\Delta t[(\Delta t)^4 + (\Delta t)^4(\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^n\|^2 + \|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^{n-1}\|^2)\\
&+\|\partial^2\eta^n\|^2 + \|d_t\xi^n\|^2 + \|d_t\xi^{n-1}\|^2]
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&\sum_{n=1}^{N-1} I_7 = \sum_{n=1}^{N-1}(\Delta t\sum_{i=1}^{n} b(t_n, t_i, u^i)\nabla\tilde{u}^i - \Delta t\sum_{i=1}^{n} b(t_n, t_i, U^i)\nabla\tilde{u}^i, \nabla(\xi^{n+1} - \xi^{n-1}))\\
&-\sum_{n=1}^{N-1}(\Delta t\sum_{i=1}^{n} b(t_n, t_i, U^i)\nabla U^i - \Delta t\sum_{i=1}^{n} b(t_n, t_i, U^i)\nabla\tilde{u}^i, \nabla(\xi^{n+1} - \xi^{n-1}))\\
&+\sum_{n=1}^{N-1}((\int_0^{t_n} b(t_n, s, u(x,s))\nabla\tilde{u}^n ds - \Delta t\sum_{i=1}^{n} b(t_n, t_i, u^i)\nabla\tilde{u}^i, \nabla(\xi^{n+1} - \xi^{n-1}))\\
&= I_7^{(1)} + I_7^{(2)} + I_7^{(3)}
\end{aligned} \tag{15}
$$

$$
I_7^{(3)} \leq C\Delta t\sum_{n=1}^{N-1}((\Delta t)^4 + \|d_t\xi^n\|^2 + \|d_t\xi^{n-1}\|^2) \tag{16}
$$

$$
\begin{aligned}
&I_7^{(1)} \leq N\Delta t|\sum_{n=1}^{N-1}((b(u^n) - b(U^n))\nabla\tilde{u}^n, \nabla(\xi^{n+1} - \xi^{n-1}))|\\
&\leq C|\sum_{n=1}^{N-1}((b(u^n) - b(U^n))\nabla\tilde{u}^n, \nabla(\xi^{n+1} - \xi^{n-1}))|
\end{aligned} \tag{17}
$$

We estimate $I_7^{(1)}$ as the same as $\sum_{n=1}^{N-1} I_3^{(1)}$:

$$
\begin{aligned}
&I_7^{(1)} \leq \epsilon(\|\xi^N\|_1^2 + \|\xi^{N-1}\|_1^2) + C(\|\xi^1\|_1^2 + \|\eta^1\|^2 + \|\xi^1\|^2) + C\Delta t\sum_{n=1}^{N-2}(\|d_t\xi^n\|^2 +\\
&\|d_t\eta^n\|^2 + \|\xi^n\|_1^2 + \|\eta^n\|^2 + \|\xi^{n+1}\|^2 + \|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n+1}\|_1^2)
\end{aligned} \tag{18}
$$

$$|I_7^{(2)}| \leq C|(b(U^{N-1})\nabla\xi^{N-1}, \nabla\xi^N) - (b(U^0)\nabla\xi^0, \nabla\xi^1)$$
$$-C\Delta t\sum_{n=1}^{N-2}\|d_tU^n\|_{L^\infty}(\|\xi^{n+1}\|_1^2 + \|\xi^n\|_1^2)|$$
$$\leq \epsilon(\|\xi^{N-1}\|_1^2 + \|\xi^N\|_1^2) + C\|\xi^1\|_1^2 + C\Delta t\sum_{n=1}^{N-2}(\|\xi^{n+1}\|_1^2 + \|\xi^n\|_1^2) \quad (19)$$

Put $I_3^{(2)}$ on the left-hand side of (8), we can get:

$$(\frac{\xi^{n+1} - 2\xi^n + \xi^{n-1}}{\Delta t^2}, \xi^{n+1} - \xi^{n-1}) + \lambda(\nabla(\xi^{n+1} - 2\xi^n + \xi^{n-1}), \nabla(\xi^{n+1} - \xi^{n-1}))$$
$$+\lambda^2\Delta t^2(\frac{\partial^2}{\partial x_1\partial x_2}(\xi^{n+1} - 2\xi^n + \xi^{n-1}), \frac{\partial^2}{\partial x_1\partial x_2}(\xi^{n+1} - \xi^{n-1}))$$
$$+(a(U^n)\nabla\xi^n, \nabla(\xi^{n+1} - \xi^{n-1}))$$
$$= \|d_t\xi^n\|^2 - \|d_t\xi^{n-1}\|^2 + \lambda(|\ \xi^{n+1} - \xi^n\ |_1^2 - |\ \xi^n - \xi^{n-1}\ |_1^2)$$
$$+\lambda^2\Delta t^4(\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^n\|^2 - \|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^{n-1}\|^2) + (a(U^n)\nabla\xi^n, \nabla\xi^{n+1})$$
$$-(a(U^{n-1})\nabla\xi^{n-1}, \nabla\xi^n) - ((a(U^n) - a(U^{n-1}))\nabla\xi^{n-1}, \nabla\xi^n)$$

Summing up the left-hand side of (8) for n from one to $N-1$, we can get:

$$\sum_{n=1}^{N-1}(\frac{\xi^{n+1} - 2\xi^n + \xi^{n-1}}{\Delta t^2}, v) + \lambda(\nabla(\xi^{n+1} - 2\xi^n + \xi^{n-1}), \nabla v)$$
$$+\lambda^2\Delta t^2(\frac{\partial^2}{\partial x_1\partial x_2}(\xi^{n+1} - 2\xi^n + \xi^{n-1}), \frac{\partial^2 v}{\partial x_1\partial x_2})$$
$$= \|d_t\xi^{N-1}\|^2 - \|d_t\xi^0\|^2 + \lambda^2\Delta t^4(\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^{N-1}\|^2 - \|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^0\|^2)$$
$$-\sum_{n=1}^{N-1}((a(U^n) - a(U^{n-1}))\nabla\xi^{n-1}, \nabla\xi^n) + \lambda(|\ \xi^N - \xi^{N-1}\ |_1^2 - |\ \xi^1 - \xi^0\ |_1^2)$$
$$+a(U^{N-1})\nabla\xi^{N-1}, \nabla\xi^N) \quad (20)$$

Refer to [4], choose $\lambda > \frac{C_1}{2}$, set $\gamma = \min\{\lambda - \frac{C_1}{2}, \frac{C_0}{4}\} > 0$, so,

$$\lambda\ |\ \xi^N - \xi^{N-1}\ |_1^2 + (a(U^{N-1})\nabla\xi^{N-1}, \nabla\xi^N) \geq \gamma(|\ \xi^N - \xi^{N-1}\ |_1^2 + |\ \xi^N + \xi^{N-1}\ |_1^2)$$
$$\geq C(\|\xi^N\|_1^2 + \|\xi^{N-1}\|_1^2) \quad (21)$$

$$\sum_{n=1}^{N-1}((a(U^n) - a(U^{n-1}))\nabla\xi^{n-1}, \nabla\xi^n) = C\Delta t\sum_{n=1}^{N-2}\|d_tU^n\|_{L^\infty}(\|\xi^{n+1}\|_1^2 + \|\xi^n\|_1^2)$$
$$\leq C\Delta t\sum_{n=1}^{N-2}(\|\xi^{n+1}\|_1^2 + \|\xi^n\|_1^2) \quad (22)$$

Summing up (8) for n from one to $N-1$ and applying the above analysis, we can get

$$\|d_t\xi^{N-1}\|^2+\lambda^2(\Delta t)^4\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^{N-1}\|^2+C(\|\xi^N\|_1^2+\|\xi^{N-1}\|_1^2)$$
$$\le C((\Delta t)^4+\|d_t\xi^0\|^2+\|\xi^1\|_1^2+\Delta t^4\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^0\|^2)+C\Delta t\sum_{n=1}^{N-1}(\|\partial^2\eta^n\|^2+\|\eta^n\|^2$$
$$+(\Delta t)^4h^{-2}\|\partial^2\eta^n\|_1^2)+C\Delta t\sum_{n=1}^{N-2}\|d_t\eta^n\|^2+\|\eta^1\|^2+\|\eta^{N-1}\|^2+C\Delta t\sum_{n=1}^{N-1}(\|d_t\xi^{n-1}\|^2$$
$$+\|d_t\xi^n\|^2+\|\xi^n\|_1^2+\|\xi^{n+1}\|_1^2+(\Delta t)^4(\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^n\|^2+\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^{n-1}\|^2)) \quad (23)$$

Combining (3) with (4),we know

$$C\Delta t\sum_{n=1}^{N-1}(\|\partial^2\eta^n\|^2+\|\eta^n\|^2)+C\Delta t\sum_{n=1}^{N-2}\|d_t\eta^n\|^2+\|\eta^1\|^2+\|\eta^{N-1}\|^2\le Ch^{2r}$$

$$C\Delta t\sum_{n=1}^{N-1}((\Delta t)^4h^{-2}\|\partial^2\eta^n\|_1^2)\le CN\Delta t(\Delta t)^4h^{-2}h^{2r-2}\le Ch^{2r}$$

Combining $Gronwall's$ inequality with (7),we can have

$$\|d_t\xi^{N-1}\|^2+(\Delta t)^4\|\frac{\partial^2}{\partial x_1\partial x_2}d_t\xi^{N-1}\|^2+\|\xi^N\|_1^2+\|\xi^{N-1}\|_1^2\le C((\Delta t)^4+h^{2r})$$

By the above inequality,we know that $\max\limits_{0\le n\le N-1}\|d_t\xi^n\|_{L^\infty}\le Q$,
So, $\max\limits_{0\le n\le N-1}\|d_tU^n\|_{L^\infty}\le 2Q$ i.e.,the inductive assumption hold when $n=N-1$.
By the above analysis, we can obtain the following result.

Theorem 3.1 *Let $u\in H_0^1(\Omega)$ and $U\in S_h$ be the solutions of the problems (1) and (6), respectively, a,b,f,u satisfy $(i)(ii)(iii),\lambda>\frac{C_1}{2}$,then,there exists a positive constant C, independent of h and Δt ,such that*

$$\max_{1\le n\le N}\{\|(U-u)^n\|_1\}\le C(h^r+(\Delta t)^2)$$

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Received: June 10, 2008