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Quadratic Spline Method for Solving Fourth Order Obstacle Problems

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Abstract

We introduce a new quadratic spline method for computing approximations to the solution a system of fourth order boundary value problems associated with obstacle, unilateral and contact problems. It is shown that the present method is of order two and gives approximations which are better than those produced by some other collocation, finite difference and spline methods. Numerical examples are presented to illustrate the applicability of the new method.

Mathematics Subject Classifications: 49J40, 65L10, 65L12

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1. Introduction

Variational inequalities theory has become an effective and powerful tool for studying obstacle, unilateral, contact problems arising in mathematical and engineering sciences including fluid flow through porous media, elasticity, transportation and economics equilibrium, optimal control, nonlinear optimization and operation research see, for example [1-11,13-18].

The area of obstacle problems arising in fluid flow through porous media and elasticity forms an important foundation for the applications of variational inequalities. It has been shown by Kikuchi and Oden [11] that the problem of equilibrium of elastic bodies in contact with a rigid frictionless foundation can be studied in the framework of variational inequalities. In a variational inequality formulation, the location of the free boundary (contact area) become an intrinsic part of the solution and no special techniques are needed to locate it. Various numerical methods are being developed and applied to find the numerical solutions of the obstacle problems, see, for example [1-11,13-18] and the references therein. In principle, the finite difference methods cannot be applied directly to solve the obstacle problems. However, If the obstacle is known, then the variational inequalities can be characterized by a system of differential equations by using the penalty function method of Lewy and Stampacchia [8]. The main computational advantage of this technique is its simple applicability for solving system of differential equations. In recent years, Al-Said et al [1-6], Khalifa and Noor [9] and Noor and Al-Said [16,17] have used such types of penalty function in solving a class of contact problems in elasticity in conjection with collocation, finite difference and spline techniques.

For the purpose of numerical experience, we consider an example of an elastic beam lying over an elastic obstacle. The formulation and the approximation of the elastic beam is very simple. However, it should be pointed out that the kind of numerical problems which occur for more complicated system will be the same. To convey an idea, we consider a system of fourth order boundary value problem of the type

$$u^{(4)} = \begin{cases} f(x), & a \leq x \leq c, \\ g(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases}$$

with the boundary and continuity conditions

$$\begin{aligned} u(a) &= u(b) = \alpha_1, \quad u''(a) = u''(b) = \alpha_2, \\ u(c) &= u(d) = \beta_1, \quad u''(c) = u''(d) = \beta_2, \end{aligned} \tag{1.1b}$$

where f and g are continuous functions on [a,b] and [c,d], respectively. The parameters r, $\alpha_i, i = 1,2$ and $\beta_i, i = 1,2$ are real constants. The possibility of using collocation method with quintic spline as a basis functions for solving (1.1) was discussed by Khalifa and Noor [9]. Their numerical results indicated that the quintic spline collocation method produced second order approximations. After this, Al-Said and Noor [3] used a second order finite difference method to compute second order approximations for the solution of (1.1). For related results, see [2-6,9,10,13,16-18] and the references therein. In the present paper, we develop a new quadratic spline method for solving the boundary value problem (1.1) over the whole interval $[a, b]$. In section 2, we derive the numerical method and briefly discuss its error analysis. Section 3 is devoted to the numerical experiments and comparison with other methods.

2. Numerical Method

For simplicity, we first develop the quadratic spline method for solving the fourth order boundary value problem

$$\begin{aligned} u^{(4)} &= g(x)u + f(x) \\ u(c) &= u(d) = \beta_1 \\ u''(c) &= u''(d) = \beta_2, \end{aligned} \tag{2.1}$$

then we use it to solve (1.1) in section 3. For this purpose we divide the interval $[c, d]$ into n equal subintervals using the grid points $x_i = c+ih$, $i = 0, 1, 2, \ldots, n$, $x_0 = c$, $x_{n+1} = d$ and $h = \dfrac{d-c}{n+1}$, where n is a positive integer.

Consider the problem of constructing a quadratic spline $S(x)$ satisfying the interpolation conditions $S(x_i) = u(x_i)$, for $i = 1, 2, \ldots, n$. Also, let $M_i = S_i^{(2)}$ for $i = 1, 2, \ldots, n$. Now for the differential equation in (2.1) at the konts x_i we may have

$$M_{i-1} - 2M_i + M_{i+1} = h^2 u_i^{(4)} + O(h^6) \tag{2.2}$$

The parameters of quadratic spline $S(x)$ satisfy the consistency relation

$$M_{i-1} + 2M_i + M_{i+1} = \frac{4}{h^2}[S_{i-1} - 2S_i + S_{i+1}] \tag{2.3}$$

for $i = 2, 3, \ldots, n-1$, see Al-Said [1] for more details.
It follows from (2.2) and (2.3) that

$$M_{i-1} = \frac{1}{h^2}[u_{i-1} - 2u_i + u_{i+1}] - \frac{h^2}{4}u_i^{(4)} + O(h^6) \tag{2.4}$$

for $i = 2, 3, \ldots, n$.

Now from (2.3) and (2.4) we have the consistency relations

$$u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} = \frac{h^4}{4}[u_{i-1}^{(4)} + 2u_i^{(4)} + u_{i+1}^{(4)}] + t_i \tag{2.5}$$

for $i = 2, 3, \ldots, n$. Equation (2.5) form a linear system of $n-2$ equations in the n unknowns u_i, $i = 1, 2, \ldots, n$. Thus, we need two more equations, one at each end of the range of integration. These equations are given by

$$5u_1 - 4u_2 + u_3 = 2u_0 - h^2 u_0'' + \frac{1}{4}h^4[\frac{2}{3}u_0^{(4)} + 2u_1^{(4)} + u_2^{(4)}] + t_1, \tag{2.6}$$

for $i = 1$, and

$$u_{n-2} - 4u_{n-1} + 5u_n = 2u_{n+1} - h^2 u_{n+1}'' + \frac{1}{4}h^4[u_{n-1}^{(4)} + 2u_n^{(4)} + \frac{2}{3}u_{n+1}^{(4)}] + t_n, \tag{2.7}$$

for $i = n$.

The local truncation errors t_i related to the consistency relations (2.5)-(2.7) are given by

$$t_i = \begin{cases} -\frac{31}{360}h^6 u_{i-1}^{(6)} + O(h^7), & \text{for } i = 1 \\ \frac{1}{12}h^6 u_i^{(6)} + O(h^7), & \text{for } 2 \le i \le n-1 \\ -\frac{31}{360}h^6 u_i^{(6)} + O(h^7), & \text{for } i = n. \end{cases} \tag{2.8}$$

Now, using a standard convergence analysis, see, for example Al-Said and Noor [4], it can be shown that our method is a second order convergent process, that is $\|\mathbf{e}\| \approx O(h^2)$, where $\mathbf{e} = (e_i)$ is the discretization error.

3. Application and Numerical Results

To illustrate the application of the spline method developed in the previous sections, we consider the fourth order obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} u^{(4)} \ge f(x), & \text{on } \Omega = [-1,1] \\ u \ge \psi(x), & \text{on } \Omega = [-1,1] \\ \left[u^{(4)} - f(x)\right][u - \psi(x)] = 0 & \text{on } \Omega = [-1,1] \\ u(-1) = u(1) = 0, \quad u''(-1) = u''(1) = 0, & \end{array}\right\},$$

where f is a given force acting on the beam and $\psi(x)$ is the elastic obstacle. Equation (3.1) describes the equilibrium configuration of an elastic beam, pulled at the ends and lying over an elastic obstacle. We study the problem (3.1) in the framework of variational inequality approach. To do so, we define the set K as

$$K = \{v : v \in H_0^2(\Omega) : v \ge \psi \text{ on } \Omega\}, \tag{3.1}$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev space, which is in fact a Hilbert space. It can be easily shown that the energy functional associated with the obstacle problem (3.1) is

$$\begin{aligned} I[v] &= \int_{-1}^{1} \left\{\frac{d^4v}{dx^4} - 2f(x)\right\} v(x)dx, \quad \text{for all } v \in H_0^2(\Omega) \\ &= \int_{-1}^{1} \left(\frac{d^2v}{dx^2}\right)^2 dx - 2\int_{-1}^{1} f(x)v(x)dx \\ &= a(v,v) - 2\langle f, v\rangle, \end{aligned} \tag{3.2}$$

where

$$a(u,v) = \int_{-1}^{1} \left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2v}{dx^2}\right) dx \tag{3.3}$$

and

$$\langle f, v\rangle = \int_{-1}^{1} f(x)v(x)dx. \tag{3.4}$$

It can be easily proved that the form $a(u, v)$ defined by (3.4) is bilinear, symmetric and positive (in fact, coercive) and the functional f defined by (3.5) is a linear continuous functional. It is well known [7,8,11] that the minimum u of the functional $I[v]$ defined by (3.3) on the closed convex set K in $H_0^2(\Omega)$ can be characterized by the variational inequality

$$a(u, v-u) \geq \langle f, v-u\rangle, \qquad \text{for all } v \in K. \tag{3.5}$$

Thus we conclude that the obstacle problem (3.1) is equivalent to solving the variational inequality problem (3.6). This equivalence has been used to study the existence of a unique solution of (3.1), see [7,8,11]. Now using the idea of Lewy and Stampacchia [8], the problem (3.6) can be written as

$$u^{(4)} + \nu\{u-\psi\}(u-\psi) = f, \tag{3.6}$$

where ψ is the obstacle function and $\nu(t)$ is a penalty function defined by

$$\nu(t) = \begin{cases} 4, & t \geq 0 \\ 0, & t < 0. \end{cases} \tag{3.7}$$

We assume that the obstacle function $\psi(x)$ is defined by

$$\psi(x) = \begin{cases} -\frac{1}{4} & \text{for } -1 \leq x \leq -\frac{1}{2} \text{ and } \frac{1}{2} \leq x \leq 1 \\ \frac{1}{4} & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}. \end{cases} \tag{3.8}$$

From (3.7) - (3.9), we obtain the following system of equations

$$u^{(4)} = \begin{cases} f, & \text{for } -1 \leq x \leq -\frac{1}{2} \text{ and } \frac{1}{2} \leq x \leq 1 \\ 1 - 4u + f, & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \end{cases} \tag{3.9}$$

with the boundary conditions

$$\begin{aligned} u(-1) &= u(-\tfrac{1}{2}) = u(\tfrac{1}{2}) = u(1) = 0 \\ u''(-1) &= u''(-\tfrac{1}{2}) = u''(\tfrac{1}{2}) = u''(1) = 0 \end{aligned} \tag{3.10}$$

and the conditions of continuity of u and u'' at $x = -\frac{1}{2}$ and $\frac{1}{2}$.

Example 3.1 In this example, we consider the system of differential equation (3.10) when $f = 0$, namely,

$$u^{(4)} = \begin{cases} 0, & \text{for } -1 \leq x \leq -\frac{1}{2} \text{ and } \frac{1}{2} \leq x \leq 1 \\ 1 - 4u, & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \end{cases} \tag{3.11}$$

with the boundary conditions

$$\begin{array}{ll} u(-1) & = u(-\frac{1}{2}) = u(\frac{1}{2}) = u(1) = 0 \\ u''(-1) & = -u''(-\frac{1}{2}) = u''(\frac{1}{2}) = -u''(1) = \epsilon \end{array} \tag{3.12}$$

where $\epsilon \to 0$. The analytical solution for this boundary value problem is

$$u(x) = \begin{cases} (-\frac{2}{3}x^3 - \frac{3}{2}x^2 - \frac{13}{12}x - \frac{1}{4})\epsilon, & \text{for } -1 \le x \le -\frac{1}{2} \\ 0.5 - \frac{1}{\phi_3}[\phi_1 \sin x \sinh x + \phi_2 \cos x \cosh x], & \text{for } -\frac{1}{2} \le x \le \frac{1}{2} \\ (-\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{13}{12}x + \frac{1}{4})\epsilon, & \text{for } \frac{1}{2} \le x \le 1 \end{cases}, \tag{3.13}$$

where $\phi_1 = \sin\frac{1}{2} \sinh\frac{1}{2}$, $\phi_2 = \cos\frac{1}{2} \cosh\frac{1}{2}$ and $\phi_3 = \cos 1 + \cosh 1$.
Example 3.2 For $f = 1$, the problem (3.10) becomes

$$u^{(4)} = \begin{cases} 1, & \text{for } -1 \le x \le -\frac{1}{2} \text{ and } \frac{1}{2} \le x \le 1 \\ 2 - 4u, & \text{for } -\frac{1}{2} \le x \le \frac{1}{2} \end{cases} \tag{3.14}$$

with the boundary conditions (3.11). The analytical solution for this boundary value problem is

$$u(x) = \begin{cases} \frac{1}{24}x^4 + \frac{1}{8}x^3 + \frac{1}{8}x^2 + \frac{3}{64}x + \frac{1}{192}, & \text{for } -1 \le x \le -\frac{1}{2} \\ 0.5 - \frac{1}{\phi_3}[\phi_1 \sin x \sinh x + \phi_2 \cos x \cosh x], & \text{for } -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{1}{24}x^4 - \frac{1}{8}x^3 + \frac{1}{8}x^2 - \frac{3}{64}x + \frac{1}{192}, & \text{for } \frac{1}{2} \le x \le 1 \end{cases}, \tag{3.15}$$

where ϕ_1, ϕ_2 and ϕ_3 are as defined in example 4.1.

The problems (3.12) and (3.15) were solved over the whole interval [-1,1] using the spline method developed in section 2 with a variety of h and ϵ values. The observed maximum errors in absolute values are given in Tables 1 and 2. From the these tables we can notice that if the stepsize h is reduced by a factor 1/2, then the errors are approximately reduced by a factor 1/4. Thus, the numerical results confirm that our present method is a second-order convergent process as predicted in section 3. These problems were also solved in [2-4,6,10] using second order finite difference and spline methods, and problem (3.12) was solved [9] using collocation method with quintic B-spline as basis functions. Some of their results are also given in Tables 1 and 2. From these tables we may notice that our present method gives better results than the others.

Table 1: Observed maximum errors for example 3.1 with $\epsilon = 10^{-6}$.

h	Our method	[3]	[4]	[6]	[9]	[10]
$\frac{1}{8}$	6.5×10^{-6}	1.4×10^{-4}	1.3×10^{-5}	7.2×10^{-6}	3.0×10^{-4}	1.9×10^{-5}
$\frac{1}{16}$	1.6×10^{-6}	3.6×10^{-5}	3.2×10^{-6}	2.2×10^{-6}	7.0×10^{-5}	4.8×10^{-6}
$\frac{1}{32}$	4.1×10^{-7}	8.9×10^{-6}	8.1×10^{-7}	5.7×10^{-7}	1.4×10^{-5}	1.2×10^{-6}

Table 2: Observed maximum errors for example 3.2.

h	Our method	[3]	[4]	[6]	[9]
$\frac{1}{12}$	6.8×10^{-6}	6.2×10^{-5}	1.2×10^{-5}	7.8×10^{-6}	8.4×10^{-6}
$\frac{1}{24}$	1.6×10^{-7}	1.6×10^{-5}	2.8×10^{-6}	1.9×10^{-6}	2.2×10^{-6}
$\frac{1}{48}$	4.2×10^{-7}	3.9×10^{-6}	6.9×10^{-7}	4.9×10^{-7}	5.4×10^{-7}

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