Numerical Method for Solving
Fuzzy Nonlinear Equations

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Abstract

In this paper, we suggest and analyze a new two-step iterative method for solving nonlinear fuzzy equations using the Harmonic mean rule. We prove that this method has quadratic convergence. The fuzzy quantities are presented in parametric form. Sever examples are given to illustrate the efficiency of the proposed method.

1   Introduction

In recent years much attention has been given to develop iterative type methods for solving nonlinear equations like $F(x) = 0$.

The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by Zadeh [13]. One of the major applications of fuzzy number arithmetic is nonlinear equations whose parameters are all or partially represented by fuzzy numbers[1, 6, 10]. Standard analytical techniques presented by Buckley and Qu in $[2-5]$. In this paper we have an adjustment on the classic Newton' method in order to accelerate the convergence or to reduce the number of operations and evaluations in each step of the iterative process.

We suggest and analyze an iterative method by using the Harmonic mean rule. This method is an implicit-type method. To implement this, we use Newton's method as predictor method and then use this method as corrector method. Several examples are given to illustrate the efficiency and advantage

of this two-steep method. In Section 2, we bring some basic definitions and results on fuzzy numbers. In Section 3 we develop a modification on Newton's method to introduce Harmonic Newton's method for solving of nonlinear real equations and quadratic convergence of this method has been proved. In Section 4 we apply the obtained results from Section 3 for solving of nonlinear fuzzy equations. The proposed algorithm is illustrated by some examples in Section 5 and a comparison with previous methods will be done, and conclusion is in Section 6.

2   Preliminaries

Definition 2.1 *A fuzzy number is set like* $u : \mathbb{R} \to I = [0, 1]$ *which satisfies,* [8, 12, 14],

1. u is upper semi-continuous,

2. $u(x) = 0$ outside some interval $[c, d]$,

3. There are real numbers a, b such that $c \leqslant a \leqslant b \leqslant d$ and

    3.1. $u(x)$ is monotonic increasing on $[c, a]$,
    3.2. $u(x)$ is monotonic decreasing on $[a, b]$,
    3.3. $u(x) = 1$, $a \leqslant x \leqslant b$.

Definition 2.2 *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \leqslant r \leqslant 1$, which satisfies the following requirements:*

1. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

2. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

3. $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leqslant r \leqslant 1$.

A popular fuzzy number is triangular fuzzy number $u = (a, b, c)$, with the membership function

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & a \leqslant x \leqslant b \\ \frac{x-c}{b-c}, & b \leqslant x \leqslant c, \end{cases}$$

where $c \neq a$, $c \neq b$ and hence

$$\underline{u}(r) = a + (c - a)r, \quad \overline{u}(r) = b + (c - b)r.$$

Let $TF(\mathbb{R})$ be the set of all triangular fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and $k > 0$ we defined addition $u + v$ and multiplication by real number $k > 0$ as

$$\underline{(u + v)}(r) = \underline{u}(r) + \underline{v}(r), \quad \overline{(u + v)}(r) = \overline{u}(r) + \overline{v}(r),$$
$$\underline{(ku)}(r) = k\underline{u}(r), \quad \overline{(ku)}(r) = k\overline{u}(r).$$

3 Harmonic Newton's method

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, that is, a real solution α, of the nonlinear equation system $F(x) = 0$, of n equations with n variables. This solution can be obtained as a fixed point of some function $G : \mathbb{R}^n \to \mathbb{R}^n$ by means of the fixed point iteration method

$$x_{k+1} = G(x_k), \quad k = 0, 1, \ldots,$$

where x_0 is the initial estimation. The best known fixed point method is the classical Newton's method, given by

$$x_{k+1} = x_k - J_F(x_k)^{-1} F(x_k), \quad k = 0, 1, 2, \ldots,$$

where $J_F(x_k)$ is the Jaccobian Matrix of the function F evaluated in x_k.

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function and α be a zero of the system of nonlinear equations $F(x) = 0$. The following result will be used describe the Newton's method and Harmonic Newton's method; see its proof in[9].

Lemma 3.1 *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on a convex set D. Then, for any $x, y \in D$, F satisfies*

$$F(y) - F(x) = \int_0^1 J_F(x + t(y - x))(y - x)dt. \tag{1}$$

Once the iterate x_k has been obtained, using (1):

$$F(y) = F(x_k) + \int_0^1 J_F(x_k + t(y - x_k))(y - x_k)dt. \tag{2}$$

If we estimate $J_F(x_k + t(y - x_k))$ in the interval $[0, 1]$ by its value in $t = 0$, that is by $J_F(x_k)$, and take $y = \alpha$, then

$$0 \approx F(x_k) + J_F(x_k)(\alpha - x_k),$$

is obtained, and a new approximation of α can be done by

$$x_{k+1} = x_k - J_F(x_k)^{-1} F(x_k),$$

what is the Classical *Newton method* (CN) for $k = 0, 1, \ldots$.
If an estimation of (2) is made by means of the trapezoidal rule and $y = \alpha$ is taken, then

$$0 \approx F(x_k) + \frac{1}{2}[J_F(x_k) + J_F(\alpha)](\alpha - x_k),$$

is obtained and a new approximation x_{k+1} of α is given by

$$x_{k+1} = x_k - 2[J_F(x_k) + J_F(x_{k+1})]^{-1} F(x_k).$$

In order to avoid the implicit problem that this equation involves, we use the $(k+1)$th iteration of Newton method in the right side,

$$x_{k+1} = x_k - 2[J_F(x_k) + J_F(z_k)]^{-1} F(x_k), \quad k = 0, 1, \ldots, \tag{3}$$

where

$$z_k = x_k - J_F(x_k)^{-1} F(x_k).$$

This method is called *Trapezoidal Newton's method* (TN).

   The Trapezoidal Newton's method can be understood as a *substitution of $J_F(x_k)$ in Newton's method by the arithmetic mean of $J_F(x_k)$ and $J_F(z_k)$.* If harmonic mean is used in stead of arithmetic one, then the following formula is obtained:

$$x_{k+1} = x_k - \frac{1}{2} J_F(z_k)^{-1} J_F(x_k)^{-1} [J_F(x_k) + J_F(z_k)] F(x_k), \quad k = 0, 1, \ldots, \tag{4}$$

that is,

$$x_{k+1} = x_k - \frac{1}{2} J_F(z_k)^{-1} [I + J_F(x_k)^{-1} J_F(z_k)] F(x_k), \quad k = 0, 1, \ldots, \tag{5}$$

where I is the identity matrix of size $n \times n$, and

$$z_k = x_k - J_F(x_k)^{-1} F(x_k).$$

This alternative of Newton's method will be called *Harmonic Newton's method* (HN). Now in the following, we satisfy the quadratical convergence of Harmonic Newton's method. Two following lemmas are technical lemmas whose proof can be found in [9].

Lemma 3.2 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function such that*

$$\|J_F(u) - J_F(v)\| \leqslant \|u - v\|$$

for any $u, v \in D$ convex set. Then there exists $\gamma > 0$ such that for any $x, y \in D$,

$$\|F(y) - F(x) - J_F(x)(y - x)\| \leqslant \frac{\gamma}{2}\|x - y\|^2.$$

Lemma 3.3 *Let $A \in L(\mathbb{R}^n)$ be nonsingular. If $E \in L(\mathbb{R}^n)$ and $\|A^{-1}\|.\|E\| \leqslant 1$, then $A + E$ is nonsingular and*

$$\|(A + E)^{-1}\| \leqslant \frac{\|A^{-1}\|}{1 - \|A^{-1}\|.\|E\|}.$$

Lemma 3.4 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function in α, where α is a solution of the system of nonsingular equations $F(x) = 0$. Let us suppose that $J_F(x)$ is continuous and $J_F(\alpha)$is nonsingular. Then the functions*

$$G(x) = x - C(x)^{-1}F(x),$$

where $C(x) = 2J_F(x)J_F(z)[J_F(x) + J_F(z)]^{-1}$ and $z = x - J_F(x)^{-1}F(x)$, is well-defined in a neighborhood of α, is differentiable and

$$J_G(\alpha) = I - J_F(\alpha)^{-1}J_F(\alpha) = 0.$$

Proof: Firstly, let us prove that $C(x)$ is nonsingular for any x in a neighborhood of α. Let β be $\beta = \|J_F(\alpha)^{-1}\|$ and ϵ be such that $0 < \epsilon < (2\beta)^{-1}$ is satisfied. By continuity of J_F in α there exists a $\delta > 0$ such that $\|J_F(x) - J_F(\alpha)\| \leqslant \epsilon$ if $\|x - \alpha\| \leqslant \delta$.

Now by the convergence of Classical Newton's method in [9] or [11], it can be assured that $\|z - \alpha\| \leqslant \delta$, then $\|J_F(z) - J_F(\alpha)\| \leqslant \epsilon$ and also it is concluded that $\|J_F(z) - J_F(x)\| \leqslant 2\epsilon$ if $\|x - \alpha\| \leqslant \delta$.

Then by using lemma (3.3), it is proved that $J_F(z)$ is nonsingular and

$$\begin{aligned}
\|J_F(z)^{-1}\| &= \|(J_F(\alpha) + (J_F(z) - J_F(\alpha)))^{-1}\| \\
&\leqslant \frac{\|J_F(\alpha)^{-1}\|}{1 - \|J_F(\alpha)^{-1}\|\|J_F(z) - J_F(\alpha)\|} \leqslant \frac{\beta}{1 - \beta\epsilon} < 2\beta,
\end{aligned}$$

and also, from Banach's Lemma it can be stated that $C(x)$ is nonsingular and

$$\begin{aligned}
\|C(x)^{-1}\| &= \|(2J_F(x)J_F(z)(J_F(x) + J_F(z))^{-1})^{-1}\| \\
&= \|\tfrac{1}{2}J_F(z)^{-1}J_F(x)^{-1}(J_F(x) + J_F(z))\| \\
&= \tfrac{1}{2}\|J_F(z)^{-1} + J_F(z)^{-1}J_F(x)^{-1}J_F(z)\| \\
&= \tfrac{1}{2}\|J_F(z)^{-1} + J_F(z)^{-1}J_F(x)^{-1}J_F(z) \\
&\quad -J_F(z)^{-1}J_F(x)^{-1}J_F(x) + J_F(z)^{-1}J_F(x)^{-1}J_F(x)\| \\
&= \|J_F(z)^{-1} + \tfrac{1}{2}J_F(z)^{-1}J_F(x)^{-1}(J_F(z) - J_F(x))\| \leqslant 2\beta + 2\epsilon\beta^2
\end{aligned}$$

as ϵ is arbitrary and β is constant, then it can be concluded $\|C(x)^{-1}\| \leqslant 2\beta$ for $\|x - \alpha\| \leqslant \delta$. So, the function $G(x)$ is well-defined in the neighborhood of α, $S = \{x : \|x - \alpha\| \leqslant \delta\}$.

Now, by differentiability of F in α, it can be assumed that δ is small enough to

$$\|F(x) - F(\alpha) - J_F(\alpha)(x - \alpha)\| \leqslant \epsilon\|x - \alpha\|, \quad \forall x \in S.$$

Then, for any $x \in S$,

$$
\begin{aligned}
\|G(x) &- G(\alpha) - (I - C(\alpha)^{-1}J_F(\alpha))(x - \alpha)\| \\
&= \|C(\alpha)^{-1}J_F(\alpha)(x - \alpha) - C(x)^{-1}F(x)\| \\
&\leqslant \|C(x)^{-1}(F(x) - F(\alpha) - J_F(\alpha)(x - \alpha))\| \\
&\quad + \|(C(x)^{-1}(C(x) - C(\alpha))(C(\alpha)^{-1}J_F(\alpha)(x - \alpha)\| \\
&\leqslant \|C(x)^{-1}\|.\|F(x) - F(\alpha) - J_F(\alpha)(x - \alpha)\| \\
&\quad + \|C(x)^{-1}\|.\|C(x) - C(\alpha)\|.\|x - \alpha\| \\
&\leqslant (2\beta\epsilon + 2\beta\epsilon)\|x - \alpha\| = 4\beta\epsilon\|x - \alpha\|,
\end{aligned}
$$

As ϵ is arbitrary and β is constant, then it can be concluded from the previous inequalities that G is differentiable in α, and also

$$J_G(\alpha) = I - C(\alpha)^{-1}J_F(\alpha) = I - J_F(\alpha)^{-1}J_F(\alpha) = 0. \qquad \square$$

To complete of discussion, in the following, we bring the proof of the quadratic convergence of Trapezoidal Newton's method. The Ostrowiski's Theorem in the following, is needed to convergence theorem.

Theorem 3.1 *Let $G : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable function in α, that is a solution of the system $x = G(x)$. Let $\{x_{k+1}\}_{k \geqslant 0}$ be the sequence of iterates obtained by means of fixed point iteration, $x_{k+1} = G(x_k), k = 0, 1, \ldots$. If the spectral radius of $J_G(\alpha)$ is lower than 1, then $\{x_k\}_{k \geqslant 0}$ converges to α.*

Proof: See the proof in [9].

Theorem 3.2 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable at each point of an open neighborhood D of $\alpha \in \mathbb{R}$, that is a solution of the system $F(x) = 0$. Let us suppose that $J_F(x)$ is continuous and nonsingular in α. Then the sequence $\{x_k\}_{k \geqslant 0}$ obtained using the iterative expression (3) converges to α and*

$$\lim_{k \to \infty} \frac{\|x_{k+1} - \alpha\|}{\|x_k - \alpha\|} = 0.$$

Moreover, if there exists $\gamma > 0$ such that

$$\|J_F(x) - J_F(\alpha)\| \leqslant \gamma\|x - \alpha\|,$$

for any x in D, then there exists a constant $M > 0$ such that

$$\|x_{k+1} - \alpha\| \leqslant M\|x_k - \alpha\|^2, \quad \forall k \geqslant k_0,$$

where k_0 depends on the initial estimation x_0.

Proof: From Lemma 3.4 we can assure that

$$G(x) = x - C(x)^{-1}F(x),$$

where $C(x)^{-1} = \frac{1}{2}J_F(z)^{-1}J_F(x)^{-1}(J_F(x) + J_F(z))$, $z = x - J_F(x)^{-1}F(x)$, is well-defined in a neighborhood of α, is differentiable in α and $J_G(\alpha) = I - J_F(\alpha)^{-1}J_F(\alpha) = 0$, and also that $\|C(x)^{-1}\| < 2\beta$, where $\beta = \|J_F(\alpha)^{-1}\|$.

If the sequence $\{x_k\}_{k \geqslant 0}$ is obtained by means of fixed point iteration on G, using Theorem 3.1 it can be concluded that $\{x_k\}_{k \geqslant 0}$ converges to α. Moreover, as G is differentiable in α,

$$\lim_{k \to \infty} \frac{\|G(x_k) - G(\alpha) - J_G(\alpha)(x_k - \alpha)\|}{\|x_k - \alpha\|} = 0,$$

but $J_G(\alpha) = 0$, so this limit is equivalent to:

$$\lim_{k \to \infty} \frac{\|G(x_k) - G(\alpha)\|}{\|x_k - \alpha\|} = \lim_{k \to \infty} \frac{\|x_{k+1} - \alpha\|}{\|x_k - \alpha\|} = 0.$$

Now, if $\|J_F(x) - J_F(\alpha)\| \leqslant \gamma\|x - \alpha\|$ for any x in a neighborhood of α, an analogous reasoning to the one made in the proof of Lemma 3.2 allows us to assure that, for any x in the neighborhood of α and from $C(x) = 2J_F(x)J_F(z)[J_F(x) + J_F(z)]^{-1}$, $C(\alpha) = J_F(\alpha)$ in Lemma 3.4,

$$\|F(x) - F(\alpha) - C(\alpha)(x-\alpha)\| = \|F(x) - F(\alpha) - J_F(\alpha)(x-\alpha)\| \leqslant \frac{1}{2}\gamma\|x - \alpha\|^2$$

So, by the convergence of Classical Newton's method, $\|J_F(z) - J_F(\alpha)\| \leqslant \gamma\|x - \alpha\|$ and $\|J_F(x) - J_F(\alpha)\| \leqslant \gamma\|x - \alpha\|$ for any x in the neighborhood of α , so is obtained that,

$$
\begin{aligned}
\|C(x) - C(\alpha)\| &= \|2J_F(x)J_F(z)(J_F(x) + J_F(z))^{-1} - J_F(\alpha)\| \\
&= \|(J_F(x) + J_F(z))^{-1}(2J_F(x)J_F(z) - J_F(\alpha)(J_F(x) + J_F(z)))\| \\
&= \|(J_F(x) + J_F(z))^{-1}(J_F(x)(J_F(z) - J_F(\alpha)) + J_F(z)(J_F(x) - J_F(\alpha))\| \\
&\leqslant \|(J_F(x) + J_F(z))^{-1}(J_F(x)\gamma\|x - \alpha\| + J_F(z)\gamma\|x - \alpha\|) \leqslant \epsilon
\end{aligned}
$$

then is concluded that

$$
\begin{aligned}
\|G(x) - G(\alpha)\| &= \|x - C(x)^{-1}F(x) - \alpha\| \\
&= \|C(x)^{-1}[F(x) - F(\alpha) - C(\alpha)(x - \alpha)] - C(x)^{-1}[C(x) - C(\alpha)](x - \alpha)\| \\
&\leqslant \|C(x)^{-1}[F(x) - F(\alpha) - C(\alpha)(x - \alpha)]\| + \|C(x)^{-1}[C(x) - C(\alpha)](x - \alpha)\| \\
&\leqslant \|C(x)^{-1}\| \cdot \|F(x) - F(\alpha) - C(\alpha)(x - \alpha)\| + \|C(x)^{-1}\| \cdot \|C(x) - C(\alpha)\| \cdot \|x - \alpha\| \\
&\leqslant \gamma\beta\|x - \alpha\|^2 + 2\gamma\beta\|x - \alpha\|^2 = 3\gamma\beta\|x - \alpha\|^2,
\end{aligned}
$$

in a neighborhood of α. Thus,

$$\|x_{k+1} - \alpha\| \leqslant M\|x_k - \alpha\|^2,$$

is satisfied with $M = 3\gamma\beta$. $\qquad\square$

4 Harmonic Newton's method for fuzzy equations

Now our aim in this section is to obtain a solution for nonlinear equation $F(x) = 0$.

The parametric form is as follows:

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}; r) = 0, \\ \overline{F}(\underline{x}, \overline{x}; r) = 0. \end{cases} \quad \forall r \in [0, 1] \tag{6}$$

Suppose that $\alpha = (\underline{\alpha}, \overline{\alpha})$ is the solution to the system (6), i.e,

$$\begin{cases} \underline{F}(\underline{\alpha}, \overline{\alpha}; r) = 0, \\ \overline{F}(\underline{\alpha}, \overline{\alpha}; r) = 0, \end{cases} \quad \forall r \in [0, 1]$$

Therefore, if $x_0 = (\underline{x}_0, \overline{x}_0)$ is an approximation solution for this system, then $\forall r \in [0, 1]$, there are $h(r), k(r)$ such that

$$\begin{cases} \underline{\alpha}(r) = \underline{x}_0(r) + h(r), \\ \overline{\alpha}(r) = \overline{x}_0(r) + k(r). \end{cases}$$

Now by using of the Taylor series of $\underline{F}, \overline{F}$ about $(\underline{x}_0, \overline{x}_0)$, then $\forall r \in [0, 1]$,

$$\begin{cases} \underline{F}(\underline{\alpha}, \overline{\alpha}; r) &= \underline{F}(\underline{x}_0, \overline{x}_0; r) + h\underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) + O(h^2 + hk + k^2) \\ &= 0, \\ \overline{F}(\underline{\alpha}, \overline{\alpha}; r) &= \overline{F}(\underline{x}_0, \overline{x}_0; r) + h\overline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\overline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) + O(h^2 + hk + k^2) \\ &= 0, \end{cases}$$

and if $\underline{x}_0$ and $\overline{x}_0$ are near to $\underline{\alpha}$ and $\overline{\alpha}$, respectively, then $h(r)$ and $k(r)$ are small enough. Let us suppose that all needed partial derivatives exists are bounded. Therefore for enough small $h(r)$ and $k(r)$, where $\forall r \in [0, 1]$, we have,

$$\begin{cases} \underline{F}(\underline{x}_0, \overline{x}_0; r) + h\underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) = 0, \\ \overline{F}(\underline{x}_0, \overline{x}_0; r) + h\overline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) + k\overline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) = 0, \end{cases}$$

and hence $h(r)$ and $k(r)$ are unknown quantities that can be obtained by solving the following equations, $\forall r \in [0, 1]$,

$$J_F(\underline{x}_0, \overline{x}_0; r) \begin{bmatrix} h(r) \\ k(r) \end{bmatrix} = \begin{bmatrix} -\underline{F}(\underline{x}_0, \overline{x}_0; r) \\ -\overline{F}(\underline{x}_0, \overline{x}_0; r) \end{bmatrix}, \tag{7}$$

where

$$J_F(\underline{x}_0, \overline{x}_0; r) = \begin{bmatrix} \underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) & \underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) \\ \overline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0; r) & \overline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0; r) \end{bmatrix}.$$

is the Jaccobian Matrix of the function $F = (\underline{F}, \overline{F})$ evaluated in $x_0 = (\underline{x}_0, \overline{x}_0)$. Hence, the next approximations for $\underline{x}(r)$ and $\overline{x}(r)$ are as follows

$$\begin{cases} \underline{x}_1(r) = \underline{x}_0(r) + h(r), \\ \overline{x}_1(r) = \overline{x}_0(r) + k(r), \end{cases}$$

for all $r \in [0, 1]$.

We can obtain approximated solution, $r \in [0, 1]$, by using the recursive scheme

$$\begin{cases} \underline{x}_{n+1}(r) = \underline{x}_n(r) + h_n(r), \\ \overline{x}_{n+1}(r) = \overline{x}_n(r) + k_n(r), \end{cases} \tag{8}$$

where $n = 1, 2, \ldots$ Analogous to (7),

$$J_F(\underline{x}_n, \overline{x}_n; r) \begin{bmatrix} h_n(r) \\ k_n(r) \end{bmatrix} = \begin{bmatrix} -\underline{F}(\underline{x}_n, \overline{x}_n; r) \\ -\overline{F}(\underline{x}_n, \overline{x}_n; r) \end{bmatrix}, \tag{9}$$

Now, let $J_F(\underline{x}_n, \overline{x}_n; r)$ be nonsingular, then from (6) recursive scheme of Newton's method is obtained as follows,

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \overline{x}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \end{bmatrix} - J_F(\underline{x}_n, \overline{x}_n; r)^{-1} \begin{bmatrix} \underline{F}(\underline{x}_n, \overline{x}_n; r) \\ \overline{F}(\underline{x}_n, \overline{x}_n; r) \end{bmatrix}. \tag{10}$$

From *Harmonic Newton's method* (HN) in Section 3, by substitution of $J_F(\underline{x}_n, \overline{x}_n; r)$ in (9) by

$$\begin{aligned} C(\underline{x}_n, \overline{x}_n; r) &= 2J_F(x; r)J_F(z; r)(J_F(x; r) + J_F(z; r))^{-1} \\ &= 2J_F(\underline{x}_n, \overline{x}_n; r)J_F(\underline{z}_n, \overline{z}_n; r)(J_F(\underline{x}_n, \overline{x}_n; r) + J_F(\underline{z}_n, \overline{z}_n; r))^{-1}, \end{aligned}$$

where

$$\begin{bmatrix} \underline{z}_n \\ \overline{z}_n \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \end{bmatrix} - J_F(\underline{x}_n, \overline{x}_n; r)^{-1} \begin{bmatrix} \underline{F}(\underline{x}_n, \overline{x}_n; r) \\ \overline{F}(\underline{x}_n, \overline{x}_n; r) \end{bmatrix},$$

and by using Lemma 3.4, $C(\underline{x}_n, \overline{x}_n; r)$ is nonsingular, then similar to (10) in Newton's method, recursive scheme for Harmonic Newton's method (HN) is obtained as follows

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \overline{x}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \end{bmatrix} - \frac{1}{2}J_F(z; r)^{-1}J_F(x; r)^{-1}(J_F(x; r) + J_F(z; r)) \begin{bmatrix} \underline{F}(\underline{x}_n, \overline{x}_n; r) \\ \overline{F}(\underline{x}_n, \overline{x}_n; r) \end{bmatrix}, \tag{11}$$

or

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \overline{x}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \end{bmatrix} - \frac{1}{2}J_F(z_n; r)^{-1}(I + J_F(x_n; r)^{-1}J_F(z_n; r)) \begin{bmatrix} \underline{F}(\underline{x}_n, \overline{x}_n; r) \\ \overline{F}(\underline{x}_n, \overline{x}_n; r) \end{bmatrix}, \tag{12}$$

where $n = 1, 2, \ldots$ and I is the identity matrix of size $n \times n$.
For initial guess, one can use the fuzzy number

$$x_0 = (\underline{x}(0), \underline{x}(1), \overline{x}(0))$$

and in parametric form

$$\underline{x}_0(r) = \underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r, \quad \overline{x}_0(r) = \overline{x}(0) + (\underline{x}(1) - \overline{x}(0))r,$$

when $\underline{x}(0) \leqslant \underline{x}(1) \leqslant \overline{x}(0)$.

Finally, in the following it is shown that, under certain conditions, Harmonic Newton's method for fuzzy equation $F(x) = 0$ is convergent and that this convergence is quadratical.

Theorem 4.1 *Let $\forall r \in [0, 1]$, the functions $\underline{F}$ and $\overline{F}$ are continuously differentiable with respect to $\underline{x}$ and $\overline{x}$. Assume that there exists $(\underline{\alpha}(r), \overline{\alpha}(r)) \in \mathbb{R}^2$ and a $\beta > 0$ such that $\|J_F(\underline{\alpha}, \overline{\alpha}; r)^{-1}\| \leqslant \beta$ and J_F will be Lipschitz continuous with respect to $\underline{x}$ and $\overline{x}$ with constant γ, then the Trapezoidal Newton's method converges to $(\underline{\alpha}, \overline{\alpha})$, and there exists a $M > 0$ such that,*

$$\|(\underline{x}_{n+1}, \overline{x}_{n+1}) - (\underline{\alpha}, \overline{\alpha})\| \leqslant M\|(\underline{x}_n, \overline{x}_n) - (\underline{\alpha}, \overline{\alpha})\|^2.$$

Proof. From Theorem 3.2, for $n = 2$, the result is concluded. $\qquad \square$

5   Numerical application

Example 5.1 *Consider the fuzzy nonlinear equation* [2]

$$(3, 4, 5)x^2 + (1, 2, 3)x = (1, 2, 3).$$

Without any loss of generality, assume that x is positive, then the parametric form of this equation is as follows

$$\begin{cases} (3 + r)\underline{x}^2(r) + (1 + r)\underline{x}(r) = (1 + r), \\ (5 - r)\overline{x}^2(r) + (3 - r)\overline{x}(r) = (3 - r), \end{cases}$$

or equality

$$\begin{cases} (3 + r)\underline{x}^2(r) + (1 + r)\underline{x}(r) - (1 + r) = 0, \\ (5 - r)\overline{x}^2(r) + (3 - r)\overline{x}(r) - (3 - r) = 0. \end{cases}$$

To obtain initial guess we use above system for $r = 0$ and $r = 1$, therefore

$$\begin{cases} 3\underline{x}^2(0) + \underline{x}(0) - 1 = 0, \\ 5\overline{x}^2(0) + 3\overline{x}(0) - 3 = 0, \end{cases} \quad \text{and} \quad \begin{cases} 4\underline{x}^2(1) + 2\underline{x}(1) - 2 = 0, \\ 4\overline{x}^2(1) + 2\overline{x}(1) - 2 = 0. \end{cases}$$
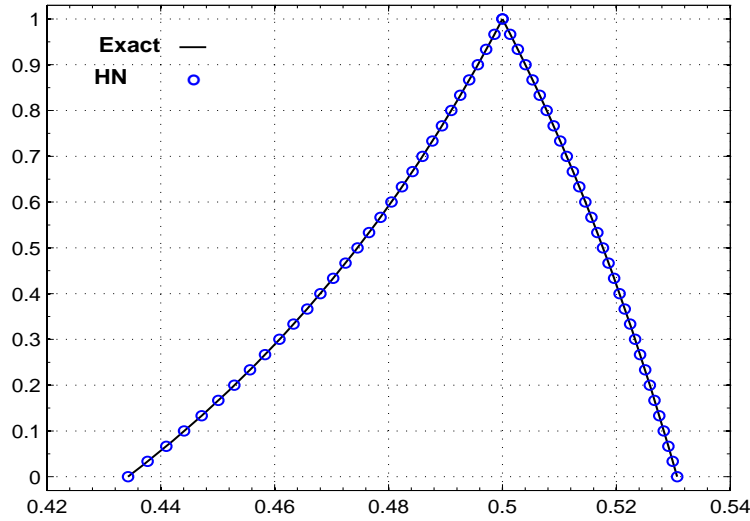
Figure 1: Solution of example 1

Consequently $\underline{x}(0) = 0.43425, \overline{x}(0) = 0.53066$ and $\underline{x}(1) = \overline{x}(1) = 0.5$. Therefore initial guess is $x_0 = (0.43425, 0.5, 0.53066)$. After 2 iterations, we obtain the solution by Harmonic Newton's method with the maximum error less than 10^{-15}, by (TN) after two iterations the maximum error less than 10^{-11} and by (CN) after two iterations the maximum error would be less than 10^{-3}. For more details see Figure 1. Now suppose x is negative, we have

$$\begin{cases} (3+r)\overline{x}^2(r) + (1+r)\underline{x}(r) - (1+r) = 0, \\ (5-r)\underline{x}^2(r) + (3-r)\overline{x}(r) - (3-r) = 0. \end{cases}$$

For $r = 0$, we have, $\underline{x}(0) \simeq -0.90705$ and $\overline{x}(0) \simeq -1.11373$, hence $\underline{x}(0) > \overline{x}(0)$, therefore negative root does not exist.

Example 5.2 *Consider fuzzy nonlinear equation* [2]

$$(1, 2, 3)x^3 + (2, 3, 4)x^2 + (3, 4, 5) = (5, 8, 13).$$

Without any loss of generality, assume that x is positive, then parametric form of this equation is as follows

$$\begin{cases} (1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) + (3+r) = (5+3r), \\ (3-r)\overline{x}^3(r) + (4-r)\overline{x}^2(r) + (5-r) = (13-5r), \end{cases}$$

or equality

$$\begin{cases} (1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) - (2+2r) = 0, \\ (3-r)\overline{x}^3(r) + (4-r)\overline{x}^2(r) - (8-4r) = 0, \end{cases}$$

Similar to Example 5.1, to obtain initial guess we use above system for $r = 0$
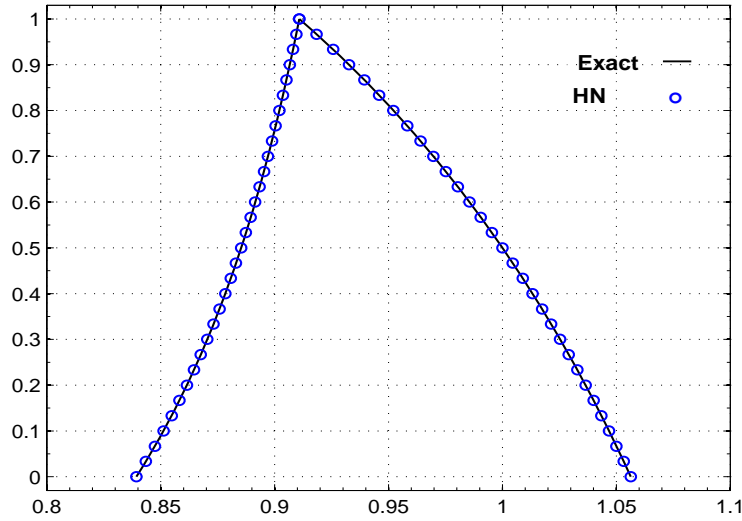
Figure 2: Positive solution of example 2

and $r = 1$, therefore

$$\begin{cases} \underline{x}^3(0) + 2\underline{x}^2(0) - 2 = 0, \\ 3\overline{x}^3(0) + 4\overline{x}^2(0) - 8 = 0, \end{cases} \quad \text{and} \quad \begin{cases} 2\underline{x}^3(1) + 3\underline{x}(1)^2 - 4 = 0, \\ 2\overline{x}^3(1) + 3\overline{x}^2(1) - 4 = 0. \end{cases}$$

Consequently $\underline{x}(0) = 0.83928, \overline{x}(0) = 1.05636$ and $\underline{x}(1) = \overline{x}(1) = 0.91082$. Therefore initial guess is $x_0 = (0.83928, 0.91082, 1.05636)$. After two iterations, we obtain the solution by Harmonic Newton's method with the maximum error less than 10^{-11}, by (TN) after two iterations the maximum error less than 10^{-8} and by (CN) after two iterations the maximum error would be less than 10^{-2}. For more details see Figure 2.

6   Conclusion

In this paper, we suggested numerical solving method for fuzzy nonlinear equations. This method is an implicit-type method. To implement this, we use Newton's method as predictor method and then use this method as corrector method. The method is discussed in detail. Several examples are given to illustrate the efficiency and advantage of this two-steep method.

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