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Preconditioned Global FOM and GMRES Methods for Solving Lyapunov Matrix Equations

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Abstract

This paper presents, a preconditioned version of global FOM and GMRES methods for solving Lyapunov matrix equations

$$AX + XA^T = -B^T B.$$

These preconditioned methods are based on the global full orthogonalization and generalized minimal residual methods. For constructing effective preconditioners, we will use ADI spiliting of above lyapunov matrix equations. Numerical experiments show that the solution of Lyapunov matrix equation can be obtained with high accuracy by using the preconditioned version of global FOM and GMRES algorithms and this version are more robust and more efficient than those without preconditioning.

Keywords: Lyapunov matrix equations, SSOR Preconditioning, ADI Preconditioning, Global FOM method, Global GMRES method

1 Introduction

Lyapunov matrix equations play a essential role in control theory [2, 3, 4, 10]. In this paper, we focus on the numerical solution of the Lyapunov matrix equations

$$AX + XA^T = -BB^T. \tag{1}$$

The necessary and sufficient condition for (1) to have a unique solution is that

$$\lambda(A) \cap \lambda(-A^T) = \emptyset,$$

where $\lambda(A)$ and $\lambda(-A^T)$ are the spectrums of A and $-A^T$, respectively. Moreover, for the symmetric right hand side, as in (1), this solution is also symmetric. There are a number of direct methods for solving the Lyapunov matrix equations (1) numerically, the most important of which are the Bartels-Stewart [1], Hessenberg-Schur [6] and the Hammarling methods [7]. But these methods is not suitable for big and sparse Lyapunov matrix equations. In [11], Y saad proposed Krylov subspace methods of Galerkin type for computing low rank solutions of large and sparse Lyapunov matrix equations. His methods are used for such large and sparse Lyapunov matrix equations which have stable matrix coefficient A, i.e., all eigenvalues of matrix A have negative real parts, and B be a vector in $\mathbb{R}^n$. The preconditioned Krylov subspace methods for large and sparse lypunov matrix equations is presented by M. Hochbruck and G. Starke [8]. They constructed SSOR and ADI(r) preconditioners for accelerating the rate of convergence of Krylov subspace methods. In addition, for solving Lyapunov matrix equations, they used coupled two-term recurrence version of QMR iterative method without look-ahead which have proposed by Freund and Nachtigal [5]. Recently K. Jbilou et al.[9] proposed global FOM and GMRES algorithms for solving matrix equations with multiple right hand sides. They extended the global GMRES method (which has presented for matrix equations) for solving Lyapunov matrix equations. In this paper we extend the global FOM method for Lyapunov matrix equations. Also, we want to propose a preconditioned version of these algorithms. We will show how we can use SSOR and ADI(r) preconditioner for these algorithms to computing symmetric solutions for the Lyapunov matrix equations. As we know, the Lyapunov matrix equations (1) can be written as a big linear system of equations

$$(I_n \otimes A + A \otimes I_n)x = b \tag{2}$$

where $\otimes$ denote the Kronecker product, $x_{ij},\ b_{ij},\ i,j = 1,...,n$ are entries of matrices X and $-BB^T$ respectively, and $x = (x_{11},...,x_{1n},...,x_{n1},...,x_{nn})^T$ and $b = (b_{11},...,b_{1n},...,b_{n1},...,b_{nn})^T$. One possibility to derive iterative methods for the solution of (1) is to take any of the well-known iterative schemes for the solution of large system (2) with the coefficient matrix

$$\mathcal{A} = I_n \otimes A + A \otimes I_n$$

and the reformulate it in terms of (1). In this manner, the ADI method [12] with respect to the spiliting of the linear system into $I \otimes A$and $A \otimes I$ leads to smiths method

$$(A - p_j I_n)X'_{j-1} = -[BB^T + X_{j-1}(A + p_j I_n)^T] \tag{3}$$

$$X_j(A - p_j I_n)^T = -[BB^T + (A + p_j I_n)X'_{j-1}]$$

with real parameters $p_j,\ j = 1,...,$ we will refer to this as the ADI method for Lyapunov matrix equations. Our purpose in this paper is to study global FOM and GMRES algorithms based on Kronecker sum formulation and to present several preconditioner for this approach theorically and numerically. Since the solution X of (1) is symmetric matrix it is of particular interest to have symmetric iterates throughout the iteration. we will investigate this topic and prove that this is the case for global FOM and GMRES methods without preconditioning, and when using (point) SSOR or ADI-type preconditioners. M. Huchbrock and G. Starke used the stationary ADI method, i.e., $p_j = p, j = 1, ...,$ to preconditioned Lyapunov matrix equations. This approach takes the form

$$A(A - pI_n)^{-1}Y(A - pI_n)^{-T} + (A - pI_n)^{-1}Y(A - pI_n)^{-T}A^T = -B^T B \quad (4)$$

$$(A - pI_n)^{-1}Y(A - p_n)^{-T} = X$$

corresponding to right preconditioning for the corresponding linear system. Therefore, we need information about the location of the eigenvalues of the matrix A to choice appropriate parameter p (or parameter sets for higher order of ADI preconditioning). Because the matrix A is relatively small compared to the size of problem, it pays to compute (or at least estimate) its eigenvalues and optimal parameters.

Throughout this paper, we use the following notations. $e_1^{(k)}$ denotes the first axis vector of dimension k, $\mathbb{E} = \mathbb{E}_{n\times n}$ denotes the vector space, on the field $\mathbb{R}$, of square matrices of dimension $n \times n$. For X and Y in $\mathbb{E}$, we define the inner product $< X, Y >_F$=tr$(X^T Y)$, where tr(Z) denotes the trace of the square matrix Z and X^T denotes the transpose of the matrix X. The associated norm is the well-known Frobenius norm denoted by $\| \cdot \|_F$. For a matrix $V \in \mathbb{E}$, the block Krylov subspace $\mathcal{K}_m(A, V)$ is the subspace generated by the columns of the matrices $V, AV, ..., A^{m-1}V$. A set of members of $\mathbb{E}$ is said to be F-Orthonormal if it is orthonormal with respect to scaler product $< \cdot \, , \, \cdot >_F$.

This paper is organized as follows. In Section 2, a brief description of the global FOM and GMRES methods for solving matrix equations are given, and The point SSOR and ADI preconditioners for solving Lyapunov matrix equations are summarized in Section 3. In Section 4 some numerical examples are tested. Finally, Section 5 summarizes the main conclusion of this paper.

2 Global FOM and GMRES Algorithms For Solving Lyapunov Matrix Equations

Global FOM and GMRES methods for matrix equations have recently presented by K. Jbilou et al. These methods are based on global oblique and orthogonal projections of the initial matrix residual onto a matrix Krylov subspace. As we know [9], the modified global Arnoldi algorithm constructs an

F-Orthonormal basis $V_1, V_2, ..., V_m$, of Krylov subspace $\mathcal{K}_m(A, V)$. This algorithm is as follows:

Algorithm 1. Modified global Arnoldi algorithm

1. Choose an $n \times p$ matrix V_1 such that $\| V \|_F = 1$.
2. For $j = 1, ..., m$ Do:
3. $\quad \tilde{V}_j = AV_j$,
4. $\quad$ For $i = 1, \ ..., j$ Do:
5. $\quad\quad h_{ij} = tr(V_i^T \tilde{V}_j)$,
6. $\quad\quad \tilde{V}_j = \tilde{V}_j - h_{ij} V_i$,
7. $\quad\quad h_{j+1,j} = \| \tilde{V}_j \|_F$
8. $\quad$ EndDo.
9. $\quad$ If $h_{j+1,j} = 0$ then stop.
10. $\quad$ Set $V_{j+1} = \tilde{V}_j / h_{j+1,j}$.
11. EndDo

Here we give a brief description of the global GMRES and FOM methods for solving Lyapunov matrix equations. For presenting these methods, first we define the following linear operator,

$$\mathcal{S} : \mathrm{IR}^{n \times n} \to \mathrm{IR}^{n \times n}$$

$$\mathcal{S}X = AX + XA^T.$$

By this definition we can see that equation (1) is equivalent with

$$\mathcal{S}X = -B^T B. \tag{5}$$

Therefore, we can obtain the solution of Lyapunov matrix equations (1), by solving matrix equations (5). Thus, the global FOM and GMRES methods for solving Lyapunov matrix equations, use Algorithm 1 for constructing an F-Orthonormal basis $V_1, V_2, ..., V_m$, of Krylov subspace

$$\mathcal{K}_m(\mathcal{S}, V) = span\{V, \mathcal{S}V, ..., \mathcal{S}^{m-1}V\}.$$

We note that $\mathcal{S}^m V = \mathcal{S}(\mathcal{S}^{m-1}V)$. Let us collect the matrices V_i constructed by the Algorithm 1 (for obtaining an F-Orthonormal basis of Krylov subspace $\mathcal{K}_m(\mathcal{S}, V)$) in the $n \times mn$ and $n \times (m+1)n$ F-Orthonormal matrices

$$\mathcal{V}_m = [V_1, V_2, ..., V_m]$$

and

$$\mathcal{V}_{m+1} = [\mathcal{V}_m, V_{m+1}],$$

respectively. Also denote by H_m the upper $m \times m$ Hessenberg matrix whose nonzero entries are the scalars h_{ij} and the $(m+1) \times m$ matrix $\tilde{H}_m$ is the same as H_m except for an additional row whose only nonzero element is $h_{m+1,m}$ in the $(m+1, m)$ position. In this paper, also we used the notation $*$ for the following product:

$$\mathcal{V}_k * \alpha = \sum_{i=1}^{k} \alpha_i V_i,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ is a vector in $\mathbb{R}^m$, and, by the same way, we set

$$\mathcal{V}_m * H_m = [\mathcal{V}_m * H_m^{(1)}, \mathcal{V}_m * H_m^{(2)}, ..., \mathcal{V}_m * H_m^{(m)}],$$

where $H_m^{(j)}$ denotes the j-th column of the matrix H_m. It can be easily seen that the relations

$$\mathcal{V}_m * (\alpha + \beta) = \mathcal{V}_m * \alpha + \mathcal{V}_m * \beta, \quad \textit{and} \quad (\mathcal{V}_m * H_m) * \alpha = \mathcal{V}_m * (H_m \alpha),$$

where $\alpha,\ \beta \in \mathbb{R}^m$, are satisfy. By using these notations, we we have the following relations which will be used later.

Proposition 1. Let $\mathcal{V}_m = [V_1, V_2, ..., V_m]$, where the $n \times k$ matrices V_i, $i = 1, ..., k$ are defined by the global Arnoldi algorithm. Then we have

$$\| \mathcal{V}_m * \alpha \|_F = \| \alpha \|_2,$$

where $\alpha \in \mathbb{R}^m$

Proof : See [9].

Theorem 1. Let $\mathcal{V}_m, H_m$ and $\tilde{H}_m$ be defined as before. Then using the product $*$, the following relations hold:

$$S\mathcal{V}_m = \mathcal{V}_m * H_m + U_{m+1},$$

and

$$S\mathcal{V}_m = \mathcal{V}_{m+1} * \tilde{H}_m,$$

where $U_{m+1} = h_{m+1,m}[0_{n\times p}, ..., 0_{n\times p}, V_{m+1}]$.

Proof : See [9].

Now, suppose that $X_0 \in \mathbb{R}^{n\times n}$, is an initial guess to the solution X and $R_0 = -B^T B - \mathcal{S}X_0 = -B^T B - AX_0 - X_0 A^T$ is associated residual. At step m, the global full othogonalization method for solving (5), constructing the new approximation X_m to the solution (5), such that

$$X_m - X_0 = Z_m \in \mathcal{K}_m(\mathcal{S}, R_0) \tag{6}$$

with the orthogonality relation

$$R_m = -B^T B - \mathcal{S}X_m \perp_F \mathcal{K}_m(\mathcal{S}, R_0) \tag{7}$$

Relation (6) implies that

$$X_m = X_0 + \mathcal{V}_m * y_m,$$

for some $y_m \in \mathrm{I\!R}^m$. Hence from the orthogonality relation (7) we have

$$< R_m, V_i >_F = 0 \quad i = 1, ..., m.$$

Thus, we have

$$< R_0 - A(\mathcal{V}_m * y_m) - (\mathcal{V}_m * y_m)A^T, V_i >_F, \quad i = 1, ..., m,$$

and this equations lead to

$$\sum_{j=1}^{m} < \mathcal{S}V_j, V_i >_F \, y_m^{(j)} = < R_0, V_i >_F = 0 \quad i = 1, ..., m. \tag{8}$$

where $y_m = (y_m^1), ..., y_m^{(m)})^T \in \mathrm{I\!R}^m$. Using proposition 1 and theorem 1 and the fact that the matrices $V_i, i = 1, ..., m$ form an F-orthogonal basis for Krylov subspace $\mathcal{K}_m(\mathcal{S}, R_0)$, the linear system (8) can be written as follows:

$$H_m y = \| R_0 \|_F \, e_1^{(m)} \tag{9}$$

where H_m is the $m \times m$ upper Hessenberg matrix produced by modified global Arnoldi algorithm for constructing an F-orthonormal basis of Krylov subspace $\mathcal{K}_m(\mathcal{S}, R_0)$. from the relation (9), we see that at step m, we have to solve only one $m \times m$ linear system to get m-dimensional vector y_m and then $X_m = X_0 + \mathcal{V}_m * y_m$. Thus, we can write the global full orthogonalization method for solving Lyapunov matrix equations as follows.

Algorithm 2. Global full orthogonalization method for solving (1).

1. Choose an initial approximate solution X_0 and compute $R_0 = -B^T B - \mathcal{S}X_0$ and $V_1 = R_0 / \| R_0 \|_F$.
2. For $j = 1, ..., m$ apply Algorithm 1 to compute the F-orthonormal basis $V_1, V_2, ..., V_m$ of $\mathcal{K}_m(S, R_0)$ and the matrix H_m.
3. Compute y_m the solution of matrix equations of $H_m y = \| R_0 \|_F \, e_1^{(m)}$.
4. Compute $X_m = X_0 + \mathcal{V}_m * y_m$.

Since the right hand-side of the Lypunov matrix equations (1) is symmetric matrix, we can present the following result:

Proposition 2. If X_0 is symmetric, then the iterates $X_m, m = 1, ...$ produced by Algorithm 2 are all symmetric.

Proof : As $-B^T B$ and X_0 are symmetric, the residual $R_0 = -B^T B - AX_0 - X_0 A^T$ and $V_1 = R_0 / \| R_0 \|_F$ are also symmetric. Then it is easy verified by induction that $V_1, V_2, ..., V_m$, constructed by Algorithm 1 with operator $\mathcal{S}$, are all symmetric. Hence, as $X_m = X_0 + \mathcal{V}_m * y_m$, it follows that the iterates

$X_m, m = 1, ...$ produced by Algorithm 2 are all symmetric $\Box$.

By similar manner, we can obtained the global generalized minimal residual method for solving matrix equations(5). Thus the global GMRES algorithm for solving the Lyapunov matrix equations (1) can be written as follows [9].

Algorithm 3. Global Generalized Minimal Residual Method for Solving (1).

1. Choose an initial approximate solution X_0 and compute $R_0 = -B^T B - \mathcal{S} X_0$ and $V_1 = R_0 / \parallel R_0 \parallel_F$.
2. For $j = 1, ..., m$ apply Algorithm 1 to compute the F-orthonormal basis $V_1, V_2, ..., V_m$ of $\mathcal{K}_m(S, R_0)$ and the matrix $\tilde{H}_m$.
3. Compute y_m the minimizer of $\| \| R_0 \|_F e_1^{(m+1)} - \tilde{H}_m y \|_2$.
4. Compute $X_m = X_0 + \mathcal{V}_m * y_m$.

It can be easily see that proposition 2 also satisfies for Algorithm 3. Also the following proposition shows that, the convergence bound for the global GMRES algorithm applied to (5) depends on the spectrum operator $\mathcal{S}$

Proposition 3. The norm of the residual produced by Algorithm 3 at step m, $m = 1, 2, ...$ satisfy the following inequality:

$$\| R_m^G \|_F \leq \frac{L_\epsilon}{2\pi\epsilon} \| R_0^G \|_F \min_{p \in P_m, p(0)=1} (\sup_{\lambda \in \Lambda_\epsilon} | p(\lambda) |),$$

where ϵ is positive number, R_m^G is the mth residual of the global GMRES algorithm and L_ϵ be the arc length of the boundary of following set

$$\Lambda_\epsilon(\mathcal{S}) = \{ z \in \mathbb{C} : \| (zI - \mathcal{S})^{-1} \| \geq \epsilon^{-1} \}.$$

Now, suppose that, the accute angle between two matrices C and D in space $\mathbb{E}$ is defined by

$$cos(C, D) = \frac{< C, D >_F}{\| C \|_F \| D \|_F}.$$

Also, let θ_m be the accute angle between R_0 and the matrix subspace $\mathcal{S}\mathcal{K}_m(S, R_0)$, and ϕ_m be the accute angle between R_0 and $Q_m R_0$, where $Q_m R_0$, is the oblique projection onto $\mathcal{S}\mathcal{K}_m(S, R_0)$, along the F-orthogonal of $\mathcal{K}_m(S, R_0)$. Thus, we can have [9]

$$\| R_m^G \|_F = \frac{1 - cos^2 \theta_m}{tan^2 \phi_m} \| R_m^F \|_F,$$

where R_m^F is the mth residual of the global FOM Algorithm 2.

3 SSOR and ADI Preconditiononig

In this section, we want to explain two type of preconditioners (which have presented by M. Hochbruck and G. Starke [8]), for accelerating the rate of convergence of global FOM and GMRES algorithms for solving Lyapunov matrix equations. The first preconditioner for Lyapunov matrix equations is well known SSOR preconditioning of the corresponding system (2), and the second preconditioner based on ADI spiliting of (1). By spiliting the matrix A according to $A = D - L - U$ into its diagonal, strictly lower triangular and strictly upper triangular part, the corresponding decomposition of matrix

$$\mathcal{A} = I_n \otimes A + A \otimes I_n$$

can be given by

$$\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{U}$$

where

$$\mathcal{D} = I_n \otimes D + D \otimes I_n,$$

$$\mathcal{L} = I_n \otimes L + L \otimes I_n,$$

and

$$\mathcal{U} = I_n \otimes U + U \otimes I_n.$$

Thus, the SSOR preconditionerning matrix is given by

$$M_{SSOR} = \frac{1}{\omega(2-\omega)}(\mathcal{D} - \omega\mathcal{L})\mathcal{D}^{-1}(\mathcal{D} - \omega\mathcal{U}).$$

As we know [8], during the evaluating the SSOR preconditioner we have to solve the following two linear systems:

$$(\mathcal{D} - \omega\mathcal{L})y = x \Leftrightarrow (D - \omega L)Y + Y(D - \omega L^T) = X \tag{10}$$

$$(\mathcal{D} - \omega\mathcal{U})y = x \Leftrightarrow (D - \omega U)Y + Y(D - \omega U^T) = X$$

Thus, for point SSOR preconditioner with $\omega \in \mathbb{R}$ we have the following theorem.

Theorem2. If we start with symmetric matrix $X_0 \in \mathbb{R}^{n\times n}$, then the global FOM and GMRES algorithms for solving Lyapunov matrix equations, using (point) SSOR preconditioning (10) with $\omega \in \mathbb{R}$ produced symmetric iterates X_m.

Proof: The proof of this theorem is easy (see [8]).

For the second type of the preconditioner, we will explain the construction of effective preconditioners based on ADI spiliting of (1). As we know the rate of convergence of ADI method, As introduced in in (4), is strongly based on choosing appropriate parameter p. As we see in the proposition 3,the convergence bound for the global GMRES method for solving Lyapunov matrix equations involve the following quantity

$$\min\{\max_{z\in\Lambda_\epsilon(\mathcal{S})} \mid p(\lambda) \mid : \quad p \in \mathcal{P}_m, \quad p(0) = 1\}. \tag{11}$$

With this, our goal to choose the preconditioner in such that a way that the quantity (11) is minimized. M. Hochbruck and G. Starke choose the parameter p in such way that the spectrum of preconditioned operator

$$S_1 : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

$$S_1 = -2p[A(A-pI_n)^{-1}X(A-pI_n)^{-T} + (A-pI_n)^{-1}X(A-pI_n)^{-T}A^T]$$

be in a small disk bound 1. From (3), the ADI iteration operator T_1 is given by

$$T_1 : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

$$T_1 = (A+pI_n)(A-pI_n)^{-1}X(A+pI_n)^T(A-pI_n)^{-T}.$$

It is easy to see that $T_1 + S = I$, where I denotes the identity operator on $\mathbb{R}^{n\times n}$. They showed that finding an optimal ADI preconditioner in the sense of (4), leads to ADI parameter problem of choosing p in such way that

$$\max\ \{\mid 1-\gamma \mid :\quad \gamma \in \lambda(S_1)\} = \max_{\lambda\in\lambda(A)} \mid \frac{\lambda+p}{\lambda-p} \mid^2,$$

is minimized. Also the similar approach to ADI preconditioning of higher degree can be derived as following. Suppose that all parameter $p_i, i = 1, ..., r$ in the polynomial $q_r(z) = (z-p_1)...(z-p_r)$ have positive real parts. In analogy to T_1, the corresponding ADI iteration T_r, defined by

$$T_r : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

$$T_rX = q_r(-A)[q_r(A)]^{-1}X[q_r(A)]^{-T}[q_r(-A)]^T,$$

and also the preconditioned operator S_r is as follows:

$$S_r : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

$$S_rX = X - q_r(-A)[q_r(A)]^{-1}X[q_r(A)]^{-T}[q_r(-A)]^T.$$

M. Hochbruck and G. Starke showed that, obtaining appropriate parameters $p_1, ..., p_r$ in such way that the preconditioned operator S_r be as close as possible to the identity operator, leads to solving the following parameter problem

$$\min_{q_r\in\mathcal{P}_r} \max_{\lambda\in\lambda(A)} \mid \frac{q_r(-\lambda)}{q_r(\lambda)} \mid^2 .$$

In this paper we only use ADI(1) and ADI(2) preconditioners. Since the dimension of the matrix A is small compare to the complexity of the overall problem here, we may assume that the eigenvalues of of A or good approximations of them are known. Then, for small r, the above parameter problem can be solved using minimization procedures. For $r = 2$, we have

$q_2 = (z - p_1)(z - p_2) = z^2 - r_1 z + r_0$. Then the corresponding parameters problem for $r = 1$ is given by

$$\min_{p \in \mathbb{R}} \max_{\lambda \in \lambda(A)} \left| \frac{\lambda + p}{\lambda - p} \right|,$$

and for $r = 2$ is given by

$$\min_{\tau_0, \tau_1 \in \mathbb{R}} \max_{\lambda \in \lambda(A)} \left| \frac{\lambda^2 + \tau_1 \lambda + \tau_0}{\lambda^2 - \tau_1 \lambda + \tau_0} \right| .$$

The following theorem states that, ADI preconditioning also has the desirable property that starting with a symmetric matrix, the global FOM and GMRES for solving Lyapunov matrix equations are all symmetric.

Theorem 3. Let the polynomial q_r have real coefficients. Then, the starting with symmetric matrix $X_0 \in \mathbb{R}^{n \times n}$, the global FOM and GMRES for solving Lyapunov matrix equations are all symmetric iterates X_m.

Proof: The proof of this theorem is easy (see [8]).

4 Numerical Experiments

In this section, we present the performance of the Algorithm 2 and Algorithm 3 with ADI(r) and SSOR preconditioners. For all the examples, we have used the stopping criterion

$$\| R_k \|_F = \| BB^T + AX_k + X_k A^T \|_F \leq 10^{-7},$$

and the maximum number of iterations allowed, is set to 1000. For all the experiments, the initial guess is $X_0 = 0_{n \times n}$. The right-hand-side matrix $-B^T B$ is chosen so that $X = (x_{ij})$ with $x_{ij} = 1, \ 1 \leq i, j \leq n$, solves equation (1). We use the matrix A is as follows:

$$A = -\text{tridiag}(-1 + \frac{p}{n+1}, 2, -1 + \frac{p}{n+1})$$

where n is the order of matrix A. The results obtained by the Algorithm 2 and Algorithm3 are reported in Tables 1-2 with $m = 3, \ p = 1$ and different values of n. Tables 1-2 show that the number of iterations and the CPU Times in the preconditioned version of the global FOM (PGLF) and the global GMRES (PGLG) algorithms is very smaller than those of the global FOM (GLF) and the global GMRES (GLG) algorithms.

Table 1. Number of iterations (and CPU Times, second) of the GL-FOM(m)

$n \setminus$	GLF	PGLF-ADI(1)	PGLF-ADI(2)	PGLF-SSOR
100	13(1.075)	6(0.37)	4(0.251)	6(0.312)
200	23(8.125)	6(3.41)	4(1.462)	6(2.325)
300	30(33.15)	6(10.2)	4(5.711)	6(7.186)
400	38(92.85)	6(20.4)	4(10.31)	6(15.53)
500	46(213.2)	6(36.5)	4(19.11)	6(29.41)
600	54(412.2)	6(47.5)	4(32.11)	6(49.38)
700	64(754.1)	6(72.5)	4(49.25)	6(76.62)
800	74(1300)	6(107)	4(55.11)	6(117.7)
900	83(2014)	6(151)	4(102.2)	6(159.5)
1000	92(3099)	6(208)	4(140.2)	6(223.5)

Table 2. Number of iterations (and CPU Times, second) of the GL-GMRES(m)

$n \setminus$	GLG	PGLG-ADI(1)	PGLG-ADI(2)	PGLG-SSOR
100	13(0.7)	6(0.44)	4(0.314)	6(0.492)
200	22(8.2)	6(2.22)	4(1.505)	6(2.415)
300	†	6(6.81)	4(4.605)	6(7.186)
400	†	6(15.1)	4(10.85)	6(15.53)
500	†	6(27.8)	4(19.87)	6(29.41)
600	†	6(46.6)	4(31.92)	6(49.38)
700	†	6(84.1)	4(49.25)	6(79.62)
800	†	6(107)	4(72.15)	6(116.2)
900	†	6(150)	4(102.8)	6(160.1)
1000	†	6(206)	4(144.5)	6(218.5)

†=no solution has been obtained after 1000 iterations by GL-FOM(m).

5 Conclusion

We have proposed the preconditioned version of global Arnoldi algorithm for solving Lyapunov matrix equations. Our preconditioning is based on the alternating direction implicit (ADI) and SSOR methods. The numerical experiments show that the solution of Lyapunov matrix equations can be obtained with high accuracy applying the preconditioned version of Algorithms 2 and

3. In addition, using the preconditioned version of Algorithms 2 and 3 reduce the computer storage and arithmetic work required.

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