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A Subclass of Quasi-Convex Functions with Respect to Symmetric Points

Aini Janteng

School of Science and Technology, Universiti Malaysia Sabah
Locked Bag No.2073, 88999 Kota Kinabalu, Sabah, Malaysia
aini_jg@ums.edu.my

Suzeini Abdul Halim

Institute of Mathematical Sciences, Universiti Malaya
50603 Kuala Lumpur, Malaysia
suzeini@um.edu.my

Abstract

Let $C_s(A, B)$ denote the class of functions f which are analytic in an open unit disc $\mathcal{D} = \{z : |z| < 1\}$ and satisfying the condition $\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in \mathcal{D}$. In this paper, we consider the class $K_s^*(A, B)$ consisting of analytic functions f and satisfying $\frac{(zf'(z))'}{(g(z)-g(-z))'} \prec \frac{1+Az}{1+Bz}$, $g \in C_s(A, B), -1 \leq B < A \leq 1$, $z \in \mathcal{D}$. The aims of paper are to determine coefficient estimates, distortion bounds and preserving property for a certain integral operator for the class $K_s^*(A, B)$.

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1 Introduction

Let $\mathcal{U}$ be the class of functions which are analytic in the open unit disc $\mathcal{D} = \{z : |z| < 1\}$ given by

$$w(z) = \sum_{k=1}^{\infty} b_k z^k$$

and satisfying the conditions

$$w(0) = 0,\ |w(z)| < 1,\ z \in \mathcal{D}.$$

Let $\mathcal{S}$ denote the class of functions f which are analytic and univalent in $\mathcal{D}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; z \in \mathcal{D}. \qquad (1)$$

Also, let $\mathcal{S}_s^*$ be the subclass of $\mathcal{S}$ consisting of functions given by (1) satisfying

$$Re \; \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathcal{D}.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in 1959. Then, Das and Singh in 1977 extend the results of Sakaguchi to other class in $\mathcal{D}$, namely convex functions with respect to symmetric points. Let $\mathcal{C}_s$ denote the class of convex functions with respect to symmetric points and satisfying the condition

$$Re \; \left\{ \frac{(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in \mathcal{D}.$$

In 1982, Goel and Mehrok introduced a subclass of $\mathcal{S}_s^*$ which were denoted by $\mathcal{S}_s^*(A, B)$.

Let denote $\mathcal{S}_s^*(A, B)$ the class of functions of the form (1) and satisfying the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \; z \in \mathcal{D}.$$

Janteng and Halim in 2008 considered the class of functions of the form (1) denoted by $\mathcal{C}_s(A, B)$ and satisfying the condition

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \; z \in \mathcal{D}.$$

In this paper, let consider $K_s^*(A, B)$ be the class of functions of the form (1) and satisfying the condition

$$\frac{2(zf'(z))'}{(g(z) - g(-z))'} \prec \frac{1 + Az}{1 + Bz}, \; g \in C_s(A, B), -1 \leq B < A \leq 1, \; z \in \mathcal{D}.$$

Obviously $K_s^*(A, B)$ is a subclass of the class quasi-convex with respect to symmetric points, $K_s^* = K_s^*(1, -1)$.

By definition of subordination it follows that $f \in K_s^*(A, B)$ if and only if

$$\frac{2(zf'(z))'}{(g(z) - g(-z))'} = \frac{1 + Aw(z)}{1 + Bw(z)} = P(z), \quad w \in \mathcal{U} \qquad (2)$$

where

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k. \tag{3}$$

We study the class $K_s^*(A,B)$ and obtain coefficient estimates, distortion bounds and invariant character of Libera transform.

2 Preliminary Results

We need the following preliminary lemmas, required for proving our results.

Lemma 2.1 *([3]) If $P(z)$ is given by (3) then*

$$|p_n| \leq (A-B). \tag{4}$$

Lemma 2.2 *([3]) Let $N(z)$ be analytic and $D(z)$ starlike in D and $N(0) = D(0) = 0$. Then*

$$\frac{\left|\left(\frac{N'(z)}{D'(z)} - 1\right)\right|}{\left|\left(A - B\frac{N'(z)}{D'(z)}\right)\right|} < 1$$

implies

$$\frac{\left|\left(\frac{N(z)}{D(z)} - 1\right)\right|}{\left|\left(A - B\frac{N(z)}{D(z)}\right)\right|} < 1, \quad z \in \mathcal{D}.$$

Theorem 2.1 *([4]) Let $f \in C_s(A,B)$, then for $n \geq 1$,*

$$|b_{2n}| \leq \frac{(A-B)}{(2n)n!2^n} \prod_{j=1}^{n-1} (A-B+2j),$$

$$|b_{2n+1}| \leq \frac{(A-B)}{(2n+1)n!2^n} \prod_{j=1}^{n-1} (A-B+2j).$$

Theorem 2.2 *([4]) If $g \in C_s(A,B)$ then $G \in C_s(A,B)$, where*

$$G(z) = \frac{2}{z}\int_0^z g(t)\,dt.$$

3 Main Results

First, we give the coefficient inequalities for the class $K_s^*(A,B)$.

Theorem 3.1 *Let $f \in K_s^*(A,B)$, then for $n \geq 1$,*

$$|a_{2n}| \leq \frac{(A-B)}{(2n)n!2^n} \prod_{j=1}^{n-1}(A-B+2j). \tag{5}$$

and

$$|a_{2n+1}| \leq \frac{(A-B)}{(2n+1)^2(n-1)!2^{n-1}} \prod_{j=1}^{n-1}(A-B+2j). \tag{6}$$

Proof.

Since $g \in \mathcal{C}_s(A,B)$, it follows that

$$2(zg'(z))' = (g(z)-g(-z))'K(z)$$

for $z \in \mathcal{D}$, with $Re\ K(z) > 0$ where $K(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ Upon equating coefficients, we obtain

$$2^2 b_2 = c_1, \quad 3(2)b_3 = c_2, \tag{7}$$

$$4^2 b_4 = c_3 + 3b_3 c_1, \quad 5(4)b_5 = c_4 + 3b_3 c_2. \tag{8}$$

For (2) and (3), we have

$$2(1+2^2a_2z+3^2a_3z^2+4^2a_4z^3+5^2a_5z^4+...+(2n)^2a_{2n}z^{2n-1}+(2n+1)^2a_{2n+1}z^{2n}+...)$$

$$\begin{aligned} = \;& 2(1+3b_3z^2+5b_5z^4+7b_7z^6+...+(2n-1)a_{2n-1}z^{2n-2}+a_{2n+1}z^{2n}+...) \\ & \bullet\ (1+p_1z+p_2z^2+...+p_{2n}z^{2n}+p_{2n+1}z^{2n+1}+...) \end{aligned}$$

Equating the coefficients of like powers of z, we have

$$2^2 a_2 = p_1, \quad 3^2 a_3 = p_2 + 3b_3 \tag{9}$$

$$4^2 a_4 = p_3 + 3b_3 p_1, \quad 5^2 a_5 = p_4 + 3b_3 p_2 + 5b_5 \tag{10}$$

$$(2n)^2 a_{2n} = p_{2n-1} + 3b_3 p_{2n-3} + 5b_5 p_{2n-5} + ... + (2n-1)b_{2n-1}p_1 \tag{11}$$

$$(2n+1)^2 a_{2n+1} = p_{2n} + 3b_3 p_{2n-2} + 5b_5 p_{2n-4} + ... + (2n-1)b_{2n-1}p_2.$$

Easily using Lemma 2.1 and (9), we get

$$|a_2| \leq \frac{A-B}{2(2)}, \quad |a_3| \leq \frac{A-B}{3(2)}.$$

Again by applying (7), (8) and followed by Lemma 2.1, we get from (10)

$$|a_4| \leq \frac{(A-B)(A-B+2)}{4(4)(2)}, \quad |a_5| \leq \frac{(A-B)(A-B+2)}{5(4)(2)}.$$

It follows that (5) hold for n=1,2. We now prove (5) using induction. Equation (11) in conjuction with Lemma 2.1 yield

$$|a_{2n}| \leq \frac{A-B}{(2n)^2}\left[1+\sum_{k=1}^{n-1}(2k+1)|b_{2k+1}|.\right] \tag{12}$$

We assume that (5) holds for k=3,4,...,(n-1). Then from (12) and Theorem 2.1, we obtain

$$|a_{2n}| \leq \frac{A-B}{(2n)^2}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right]. \tag{13}$$

In order to complete the proof, it is sufficient to show that

$$\frac{A-B}{(2m)^2}\left[1+\prod_{k=1}^{m-1}\frac{A-B}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$= \frac{A-B}{(2m)m!2^m}\prod_{j=1}^{m-1}(A-B+2j), \quad (m=3,4,...,n). \tag{14}$$

(14) is valid for $m = 3$.

Let us suppose that (14) is true for all m, $3 < m \leq (n-1)$. Then from (13)

$$\frac{A-B}{(2n)^2}\left[1+\prod_{k=1}^{n-1}\frac{A-B}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$\begin{aligned}
&= \left(\frac{n-1}{n}\right)^2\left(\frac{A-B}{(2(n-1))^2}\left(1+\sum_{k=1}^{n-2}\frac{A-B}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right)\right) \\
&\quad + \frac{A-B}{(2n)^2}\frac{A-B}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j) \\
&= \left(\frac{n-1}{n}\right)^2\frac{A-B}{2(n-1)(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)+\frac{A-B}{(2n)^2}\frac{A-B}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j) \\
&= \frac{A-B}{2n^2(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\left(\frac{A-B+2(n-1)}{2}\right) \\
&= \frac{A-B}{(2n)n!2^n}\prod_{j=1}^{n-1}(A-B+2j)
\end{aligned}$$

Thus, (14) holds for $m = n$ and hence (5) follows. Similarly we can prove (6).

Theorem 3.2 *Let $f \in K_s^*(A,B)$, then for $|z| = r$, $0 < r < 1$,*

$$\frac{1}{r}\int_0^r \frac{1-t^2}{(1+t^2)^2}\frac{(1-At)}{(1-Bt)}\,dt \leq |f'(z)| \leq \frac{1}{r}\int_0^r \frac{1+t^2}{(1-t^2)^2}\frac{(1+At)}{(1+Bt)}\,dt. \tag{15}$$

Proof.

Put $h(z) = \frac{g(z)-g(-z)}{2}$. Then from (2), we obtain

$$|(zf'(z))'| = |h'(z)| \left|\frac{1+Aw(z)}{1+Bw(z)}\right|. \tag{16}$$

Since h is odd and starlike, it follows that (see [2])

$$\frac{1-r^2}{(1+r^2)^2} \leq |h'(z)| \leq \frac{1+r^2}{(1-r^2)^2}. \tag{17}$$

Furthermore, for $w \in \mathcal{U}$, it can also be easily established that

$$\frac{1-Ar}{1-Br} \leq \left|\frac{1+Aw(z)}{1+Bw(z)}\right| \leq \frac{1+Ar}{1+Br}. \tag{18}$$

Next, applying results (17) and (18) in (16) we obtain

$$\frac{1-r^2}{(1+r^2)^2}\frac{1-Ar}{1-Br} \leq |(zf'(z))'| \leq \frac{1+r^2}{(1-r^2)^2}\frac{1+Ar}{1+Br}. \tag{19}$$

Finally, setting $|z| = r$, and integrating (19) gives our result.

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