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Starlike Functions of Complex Order

Aini Janteng

School of Science and Technology
Universiti Malaysia Sabah, Locked Bag No. 2073
88999 Kota Kinabalu, Sabah, Malaysia
aini_jg@ums.edu.my

Abstract

Let $\mathcal{H}$ denote the class of functions f which are harmonic and univalent in the open unit disc $D = \{z : |z| < 1\}$. This paper defines and investigates a family of complex-valued harmonic functions that are orientation preserving and univalent in $\mathcal{D}$ and are related to the functions starlike of complex order. The author obtain coefficient conditions, growth result, extreme points, convolution and convex combinations.

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1 Introduction

A continuous complex-valued function $f = u+iv$ defined in a simply connected complex domain E is said to be harmonic in E if both u and v are real harmonic in E. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = Re\ (U)$ and $v = Im\ (V)$. Then

$$f(z) = h(z) + \overline{g(z)}$$

where h and g are, respectively, the analytic functions $(U+V)/2$ and $(U-V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by

$$J_f = |h'(z)|^2 - |g'(z)|^2.$$

The mapping $z \mapsto f(z)$ is orientation preserving and locally one-to-one in E if and only if $J_f > 0$ in E. The function $f = h + \overline{g}$ is said to be harmonic univalent in E if the mapping $z \mapsto f(z)$ is orientation preserving, harmonic and one-to-one in E. We call h the analytic part and g the co-analytic part of $f = h + \overline{g}$.

Let $\mathcal{H}$ denote the class of functions $f = h+\overline{g}$ which are harmonic and univalent in $\mathcal{D}$ the unit disc with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

There have been rigorous works conducted on the class of complex harmonic functions $\mathcal{H}$. In [2], Sheil-Small and Clunie obtained properties for functions in this class. Since then, there have been other subclasses of $\mathcal{H}$ that were developed. These include the class of harmonic functions starlike in the unit disc $\mathcal{D}$ (see Jahangiri [4]) and convex harmonic functions in $\mathcal{D}$ (see Kim et al. [5]). Other related works also appear in [1], [3] and [8]. Silverman in [7] formed the class $\overline{\mathcal{H}}$, a subclass of $\mathcal{H}$ which consists harmonic functions with negative coefficients. The class $\overline{\mathcal{H}}$ is defined below.

Let $\overline{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = -\sum_{n=2}^{\infty} |b_n| z^n. \tag{2}$$

Another interesting class of functions is the class of functions starlike of order $b(b \in \mathcal{C}\backslash\{0\})$, first introduced by Nasr and Aouf in [6]. Wiatrowski [9] introduced the class of functions which are convex of order $b(b \in \mathcal{C}\backslash\{0\})$. The authors, by combining defined the new class of functions as follows :

Definition 1.1 *Let $f \in \mathcal{H}$. Then $f \in \mathcal{HS}^{\star}(b,\beta)$ if and only if it satisfies*

$$\left| \frac{1}{b} \left[\frac{z f'(z)}{z' f(z)} - 1 \right] \right| < \beta. \tag{3}$$

for $b \in \mathcal{C}\backslash\{0\}, 0 < \beta \leq 1,\ z' = \frac{\partial}{\partial\theta}(z = re^{i\theta}),\ f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta})),\ 0 \leq r < 1$ and $0 \leq \theta < 2\pi$,

Also, let $\overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta) = \mathcal{HS}^{\star}(b,\beta) \cap \overline{H}$.

In this paper, the author is motivated to determine properties of this new class which include coefficients results, growth bounds, extreme points, convolution properties and convex combinations.

2 Results

The results begin with a necessary and sufficient condition for functions in $\mathcal{HS}^\star(b,\beta)$.

Theorem 2.1 *Let $f = h+\overline{g}$ with h and g of the form (2). Then $f \in \overline{\mathcal{HS}}^\star(b,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[(n-1)+\beta|b|]|a_n| + \sum_{n=2}^{\infty}[(n+1)+\beta|b|]|b_n| \leq \beta|b|. \tag{4}$$

Proof. Suppose that $f \in \overline{\mathcal{HS}}^\star(b,\beta)$. Let $w(z)$ be defined by

$$w(z) = \frac{zf'(z)}{z'f(z)} - 1 = \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1.$$

Then from (3), we obtain the following inequality:

$$|w(z)| = \left|\frac{-\sum_{n=2}^{\infty}(n-1)|a_n|z^n + \sum_{n=2}^{\infty}(n+1)|b_n|\bar{z}^n}{z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=2}^{\infty}|b_n|\bar{z}^n}\right| < \beta|b|, \quad (z \in \mathcal{D}).$$

Since $|z| = r(0 \leq r < 1)$ and $f \in \overline{\mathcal{HS}}^\star(b,\beta)$, we obtain

$$\frac{\sum_{n=2}^{\infty}(n-1)|a_n|r^{n-1} + \sum_{n=2}^{\infty}(n+1)|b_n|r^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n|r^{n-1} - \sum_{n=2}^{\infty}|b_n|r^{n-1}} < \beta|b|. \tag{5}$$

Now letting $r \to 1^-$ through real values in (5), we then have

$$\sum_{n=2}^{\infty}(n-1)|a_n| + \sum_{n=2}^{\infty}(n+1)|b_n| \leq \beta|b|\left(1 - \sum_{n=2}^{\infty}|a_n| - \sum_{n=2}^{\infty}|b_n|\right). \tag{6}$$

Thus, (6)leads us to the desired assertion (4) of Theorem 2.1.

Conversely, by applying the hypothesis (4) and letting $|z| = r(0 \leq r < 1)$, we find from (3) that

$$\begin{aligned}
&|(zh'(z) - \overline{zg'(z)}) - (h(z) + \overline{g(z)})| - \beta|b||h(z) + \overline{g(z)}| \\
&= \Big|-\sum_{n=2}^{\infty}(n-1)\ |a_n|\ z^n + \sum_{n=2}^{\infty}(n+1)\ |b_n|\ \bar{z}^n\Big| \\
&\quad -\beta|b|\Big|z - \sum_{n=2}^{\infty}\ |a_n|\ z^n - \sum_{n=2}^{\infty}\ |b_n|\ \bar{z}^n\Big| \\
&< \sum_{n=2}^{\infty}[(n-1)+\beta|b|]\ |a_n| + \sum_{n=2}^{\infty}[(n+1)+\beta|b|]\ |b_n| - \beta|b| \\
&\leq 0, \quad \textit{by } (4). \quad \square
\end{aligned}$$

Hence, $f \in \overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$ which completes the proof of Theorem 2.1.

The harmonic functions

$$f(z) = z - \sum_{n=2}^{\infty} \frac{\beta|b|}{n-1+\beta|b|} x_n z^n - \sum_{n=2}^{\infty} \frac{\beta|b|}{n+1+\beta|b|} \overline{y}_n \overline{z}^{\,n}, \tag{7}$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=2}^{\infty} |y_n| = 1$, show that the coefficient bound given in Theorem 2.1 is sharp.

The functions of the form (7) are in $\mathcal{H}\mathcal{S}^{*}(b,\beta)$ since

$$\sum_{n=2}^{\infty} \frac{n-1+\beta|b|}{\beta|b|}|a_n| + \sum_{n=2}^{\infty} \frac{n+1+\beta|b|}{\beta|b|}|b_n| = \sum_{n=2}^{\infty} |x_n| + \sum_{n=2}^{\infty} |y_n| = 1.$$

The growth result for functions in $\overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$ is discussed in the following theorem.

Theorem 2.2 *If $f \in \overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$ then*

$$|f(z)| \le r + \frac{\beta|b|}{1+\beta|b|} r^2, \quad |z| = r < 1$$

and

$$|f(z)| \ge r - \frac{\beta|b|}{1+\beta|b|} r^2, \quad |z| = r < 1.$$

Proof. Let $f \in \overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$. Taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\le r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\ r^n \\
&\le r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\ r^2 \\
&= r + \frac{\beta|b|}{1+\beta|b|} \sum_{n=2}^{\infty} \left(\frac{1+\beta|b|}{\beta|b|}|a_n| + \frac{1+\beta|b|}{\beta|b|}|b_n| \right)\ r^2 \\
&\le r + \frac{\beta|b|}{1+\beta|b|} \sum_{n=2}^{\infty} \left(\frac{n-1+\beta|b|}{\beta|b|}|a_n| + \frac{n+1+\beta|b|}{\beta|b|}|b_n| \right)\ r^2 \\
&\le r + \frac{\beta|b|}{1+\beta|b|}\ r^2
\end{aligned}$$

and

$$
\begin{aligned}
|f(z)| &\geq r - \sum_{n=2}^{\infty}(|a_n| + |b_n|) \ r^n \\
&\geq r - \sum_{n=2}^{\infty}(|a_n| + |b_n|) \ r^2 \\
&= r - \frac{\beta|b|}{1+\beta|b|} \sum_{n=2}^{\infty}\left(\frac{1+\beta|b|}{\beta|b|}|a_n| + \frac{1+\beta|b|}{\beta|b|}|b_n|\right) \ r^2 \\
&\geq r - \frac{\beta|b|}{1+\beta|b|} \sum_{n=2}^{\infty}\left(\frac{n-1+\beta|b|}{\beta|b|}|a_n| + \frac{n+1+\beta|b|}{\beta|b|}|b_n|\right) \ r^2 \\
&\geq r - \frac{\beta|b|}{1+\beta|b|} \ r^2. \quad \square
\end{aligned}
$$

Next, we determine the extreme points of closed hulls of $\overline{\mathcal{H}}\mathcal{S}^\star(b,\beta)$ denoted by $clco\overline{\mathcal{H}}\mathcal{S}^\star(b,\beta)$.

Theorem 2.3 *$f \in clco\overline{\mathcal{H}}\mathcal{S}^\star(b,\beta)$ if and only if $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$ where*

$$h_1(z) = z, \ h_n(z) = z - \frac{\beta|b|}{n-1+\beta|b|} \ z^n \ (n = 2, 3, ...),$$

$$g_n(z) = z - \frac{\beta|b|}{n+1+\beta|b|} \ \bar{z}^{\ n} \ (n = 2, 3, ...),$$

$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \ X_n \geq 0$ and $Y_n \geq 0$.

Proof. For h_n and g_n as given above, we may write

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) \\
&= \sum_{n=1}^{\infty}(X_n + Y_n)z - \sum_{n=2}^{\infty}\frac{\beta|b|}{n-1+\beta|b|}X_n z^n - \sum_{n=2}^{\infty}\frac{\beta|b|}{n+1+\beta|b|}Y_n \bar{z}^{\ n} \\
&= z - \sum_{n=2}^{\infty}\frac{\beta|b|}{n-1+\beta|b|}X_n z^n - \sum_{n=2}^{\infty}\frac{\beta|b|}{n+1+\beta|b|}Y_n \bar{z}^{\ n}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\frac{n-1+\beta|b|}{\beta|b|}|a_n| + \sum_{n=2}^{\infty}\frac{n+1+\beta|b|}{\beta|b|}|b_n| \\
&= \sum_{n=2}^{\infty}\frac{n-1+\beta|b|}{\beta|b|}\left(\frac{\beta|b|}{n-1+\beta|b|}X_n\right) \\
&\quad + \sum_{n=2}^{\infty}\frac{n+1+\beta|b|}{\beta|b|}\left(\frac{\beta|b|}{n+1+\beta|b|}Y_n\right)
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{n=2}^{\infty} X_n + \sum_{n=2}^{\infty} Y_n \\
&= 1 - X_1 - Y_1 \\
&\leq 1.
\end{aligned}$$

Therefore $f \in clco\overline{\mathcal{H}}\mathcal{S}^{\star}(b,\beta)$.

Conversely, suppose that $f \in clco\overline{\mathcal{H}}\mathcal{S}^{\star}(b,\beta)$. Set

$$X_n = \frac{n-1+\beta|b|}{\beta|b|}|a_n| \ , (n = 2,3,4,...),$$

and

$$Y_n = \frac{n+1+\beta|b|}{\beta|b|}|b_n| \ , (n = 2,3,4,...),$$

where $\sum_{n=1}^{\infty}(X_n + Y_n) = 1$. Then

$$\begin{aligned}
f(z) &= h(z) + \overline{g(z)} \\
&= z - \sum_{n=2}^{\infty} |a_n| z^n - \sum_{n=2}^{\infty} |b_n| \bar{z}^{\,n} \\
&= z - \sum_{n=2}^{\infty} \frac{\beta|b|}{n-1+\beta|b|} X_n z^{\,n} - \sum_{n=2}^{\infty} \frac{\beta|b|}{n+1+\beta|b|} Y_n \bar{z}^{\,n} \\
&= z + \sum_{n=2}^{\infty} (h_n(z) - z) X_n + \sum_{n=2}^{\infty} (g_n(z) - z) Y_n \\
&= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n). \quad \square
\end{aligned}$$

For harmonic functions $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=2}^{\infty}|b_n|\bar{z}^{\,n}$ and $F(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=2}^{\infty}|B_n|\bar{z}^{\,n}$, we define the convolution of f and F as

$$(f \star F)(z) = z - \sum_{n=2}^{\infty} |a_n A_n| z^n - \sum_{n=2}^{\infty} |b_n B_n| \bar{z}^{\,n}. \tag{8}$$

In the next theorem, we examine the convolution properties of the class $\overline{\mathcal{H}}\mathcal{S}^{\star}(b,\beta)$.

Theorem 2.4 *For $0 < \alpha \leq \beta \leq 1$, let $f \in \overline{\mathcal{H}}\mathcal{S}^{\star}(b,\beta)$ and $F \in \overline{\mathcal{H}}\mathcal{S}^{\star}(b,\alpha)$. Then $(f \star F) \in \overline{\mathcal{H}}\mathcal{S}^{\star}(b,\beta) \subset \overline{\mathcal{H}}\mathcal{S}^{\star}(b,\alpha)$.*

Proof. Write $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=2}^{\infty}|b_n|\bar{z}^{\,n}$ and $F(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=2}^{\infty}|B_n|\bar{z}^{\,n}$. Then the convolution of f and F is given by (8).

Note that $|A_n| \leq 1$ and $|B_n| \leq 1$ since $F \in \overline{\mathcal{H}}\mathcal{S}^{\star}(b,\alpha)$. Then we have

$$\sum_{n=2}^{\infty}[n-1+\beta|b|]|a_n||A_n| + \sum_{n=2}^{\infty}[n+1+\beta|b|]|b_n||B_n|$$

$$\le \sum_{n=2}^{\infty}[n-1+\beta|b|]|a_n| + \sum_{n=2}^{\infty}[n+1+\beta|b|]|b_n|.$$

Therefore, $(f \star F) \in \overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta) \subset \overline{\mathcal{H}\mathcal{S}}^{\star}(b,\alpha)$ since the right hand side of the above inequality is bounded by $\beta|b|$ while $\beta|b| \le \alpha|b|$. □

Now, we determine the convex combination properties of the members of $\overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$.

Theorem 2.5 *The class* $\overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$ *is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$, suppose that $f_i \in \overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$ where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}|z^n - \sum_{n=2}^{\infty} |b_{n,i}|\bar{z}^{\,n}.$$

For $\sum_{i=1}^{\infty} c_i = 1,\ 0 \le c_i \le 1$, the convex combinations of f_i may be written as

$$\begin{aligned}
\sum_{i=1}^{\infty} c_i f_i(z) &= c_1 z - \sum_{n=2}^{\infty} c_1|a_{n,1}|z^n - \sum_{n=2}^{\infty} c_1|b_{n,1}|\bar{z}^{\,n} - c_2 z - \sum_{n=2}^{\infty} c_2|a_{n,2}|z^n - \sum_{n=2}^{\infty} c_2|b_{n,2}|\bar{z}^{\,n} ... \\
&= z\sum_{i=1}^{\infty} c_i - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i|a_{n,i}|\right) z^n - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i|b_{n,i}|\right)\bar{z}^{\,n} \\
&= z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i|a_{n,i}|\right) z^n - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i|b_{n,i}|\right)\bar{z}^{\,n}.
\end{aligned}$$

Next, consider

$$\sum_{n=2}^{\infty}\left([n-1+\beta|b|]\left|\sum_{i=1}^{\infty} c_i|a_{n,i}|\right|\right) + \sum_{n=2}^{\infty}\left([n+1+\beta|b|]\left|\sum_{i=1}^{\infty} c_i|b_{n,i}|\right|\right)$$

$$\begin{aligned}
&= c_1\sum_{n=2}^{\infty}[n-1+\beta|b|]|a_{n,1}| + ... + c_m\sum_{n=2}^{\infty}[n-1+\beta|b|]|a_{n,m}| + ... \\
&\quad + c_1\sum_{n=2}^{\infty}[n+1+\beta|b|]|b_{n,1}| + ... + c_m\sum_{n=2}^{\infty}[n+1+\beta|b|]|b_{n,m}| + ... \\
&= \sum_{i=1}^{\infty} c_i\left\{\sum_{n=2}^{\infty}[n-1+\beta|b|]|a_{n,i}| + \sum_{n=2}^{\infty}[n+1+\beta|b|]|b_{n,i}|\right\}.
\end{aligned}$$

Now, $f_i \in \overline{\mathcal{H}\mathcal{S}}^{\star}(b,\beta)$, therefore from Theorem 2.1, we have

$$\sum_{n=2}^{\infty}[n-1+\beta|b|]|a_{n,i}| + \sum_{n=2}^{\infty}[n+1+\beta|b|]|b_{n,i}| \le \beta|b|\ .$$

Hence

$\sum_{n=2}^{\infty}\left([n-1+\beta|b|]\left|\sum_{i=1}^{\infty}c_i|a_{n,i}|\right|\right)+\sum_{n=2}^{\infty}\left([n+1+\beta|b|]\left|\sum_{i=1}^{\infty}c_i|b_{n,i}|\right|\right)$

$\leq \beta|b|\sum_{i=1}^{\infty}c_i$

$=\beta|b|.$

By using Theorem 2.1 again, we have $\sum_{i=1}^{\infty}c_i f_i \in \overline{\mathcal{H}}\mathcal{S}^{\star}(b,\beta)$. □

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