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A Control Trajectory Problem: Continuous Systems

M. Rachik[1], A. Abdelhak[2] and E. Labriji[1]

[1]University Hassan II-Mohammedia, Faculty of Sciences
Ben M'sik, Dept. of Mathematics, Casablanca, Morroco
m_rachik@yahoo.fr
labriji@math.net

[2]University Ibn Tofail, Faculty of Sciences
Departement of Mathematics, Kénitra, Morroco
abdelhak.a@gmail.com

Abstract

In this paper we consider a fundamental control problem. Our aim is not to determine the control which steers the system to a desired final state at time T, but to investigate, under some hypothesis, the input witch makes the trajectory of the system, along the interval of time $[0,T]$, to be exactly equal to a desired given one. To resolve the problem, we use a state space technique (see [1, 2, 3]), generally used in the analysis and control of hereditary systems. We also study the regional aspect of the problem.

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1 Introduction

Consider the linear system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t)\ ,\ t \in [0,T] \\ x(0) &= x_0, \end{cases} \tag{1}$$

where A is the generator of a strongly continuous semi-group $(R(t))_{t\geq 0}$ on the Hilbert space X, $B \in \mathcal{L}(U,X)$, where U is a Hilbert space.

Given a desired trajectory $x_d(.) \in L^2(0,T;X)$, we investigate [1] the control law u, having a minimal norm, which allows the state $x(t)$ to coincide with $x_d(t)$ along the interval of time $[0,T]$, i.e.,

$$\begin{cases} x(.,x_0,u) & = & x_d(.) \text{ on } L^2(0,T;X) \\ ||u|| & = & min\{||v|| : x(.,x_0,v) = x_d\} \end{cases},$$

where $x(.,x_0,v)$ is the state of system 1 corresponding to the command v.

1.1 State space technique

We consider the strongly continuous semi-group $(S(t))_{t\geq 0}$ defined on the Hilbert space $Y = L^2(-T,0;X)$ by

$$(S(t)y)(r) = \begin{cases} y(t+r) & \text{if} \quad r \in [-T,-t] \\ 0 & \text{if} \quad r \in]-t,0] \end{cases}, \quad \text{pour } t \leq T$$

and

$$(S(t)y)(r) = 0; \forall r \in [-T,0] \ , \ \forall t > T.$$

The generator of $(S(t))_{t\geq 0}$ is the operator $D = \frac{d}{ds}$ with domaine $Dom(D) = \{y \in W^{1,2}(-T,0;X) : y(0) = 0\}$. Let $F \in \mathcal{L}(X,Y)$ be the operator defined by

$$(Fx)(r) = x \ , \ \forall x \in X \ , \ \forall r \in [-T,0].$$

For every $x_0 \in X$ and all control $u \in L^2(0,T;U)$, we define on Y, the following evolution system

$$y(t,x_0,u) = S(t)(Fx_0) - D\int_0^t S(t-r)Fx(r,x_0,u)dr \ , \ t \in [0,T], \qquad (2)$$

where $x(.,x_0,u)$ is the solution of equation (1) corresponding to the control u.

Remark 1.1 *[4],[3]*
i) For every $t \leq T$

$$(y(t,x_0,u))(r) = \begin{cases} x_0 & \textit{if} \quad r \in [-T,-t] \\ x(t+r,x_0,u) & \textit{if} \quad r \in]-t,0], \end{cases}$$

hence, $(y(T,x_0,u))(r) = x(T+r,x_0,u) \ , \ \forall r \in [-T,0].$
ii) For every $t \leq T$

$$(-D\int_0^t S(t-r)Fx(r,x_0,u)dr)(s) = \begin{cases} x(t+s,x_0,u) & \textit{if} \quad s \in]-t,0] \\ 0 & \textit{if} \quad s \in [-T,-t] \end{cases}$$

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The introduction of the state $y(., x_0, u)$ allows us to transform the control trajectory problem to a standard controllability problem on the space Y. Indeed, to determine a control u such that $x(., x_0, u) = x_d(.)$ is equivalent to find a control u which satisfy $y(T) = y_d$, where $y_d \in Y$ is the desired state defined by $y_d(.) = x_d(. + T)$.

The input u acts on the signal $y(., x_0, u)$ in a non standard form, this suggest us to define an other evolution equation on the state $Z = X \times Y$. Precisely, we consider the system defined on the Hilbert space $Z = X \times Y$, by

$$z(t) = U(t)z_0 + \int_0^t U(t-r)Lu(r)dr \ , \ \forall t \in [0, T]. \tag{3}$$

where $L = (B, 0)^T$, $z_0 = (x_0, Fx_0)^T$ and $(U(t))_{t \geq 0}$ the strongly continuous semi-group defined by

$$U(t) = \begin{pmatrix} U_0(t) & 0 \\ U_1(t) & U_2(t) \end{pmatrix} \ ; \ \forall t \geq 0 \tag{4}$$

where

$$U_0(t) = R(t), \ U_2(t) = S(t) \text{ and } U_1(t) = -D\int_0^t S(t-r)FR(r)dr.$$

Remark 1.2 *We verify easily that for all $x_0 \in X$ and all control $u \in L^2(0, T; U)$, we have*

$$z(., x_0, u) = (x(., x_0, u), y(., x_0, u)), \tag{5}$$

where $z(., x_0, u)$ is the solution of (3).

1.2 Fundamental results

Define the operator H by

$$\begin{array}{rccl} H: & L^2(0, T; U) & \to & Y \\ & u & \to & p_2 \int_0^T U(T-r)Lu(r)dr \end{array}$$

where p_2 is the operator

$$\begin{array}{rccl} p_2: & Z & \to & Y \\ & (x, y) & \to & y \end{array}$$

and $(U(t))_{t \geq 0}$ is the strongly continuous semi group defined by (4).

Remark 1.3 *the operator H is bounded and have an adjoint operator given by*

$$\begin{array}{rccl} H^*: & Y & \to & L^2(0, T; U) \\ & y & \to & B^* \int_{-T}^0 R^*(T - . + r)y(r)dr. \end{array}$$

Let's consider the Hilbert space

$$F_0 = \overline{Im\,H} = (Ker\,H^*)^\perp \tag{6}$$

and the semi norm defined on Y by

$$||f||_F = ||H^* f||_{L^2(0,T;U)}$$

The corresponding scalar product to this semi norm is

$$<< f, g >>=< H^* f, H^* g > \ \forall f, g \in Y.$$

Remark 1.4 *We easily establish that*
i) $||.||_F$ *is a norm on* F_0.
ii) $(HH^*)(Y) \subset F_0$.
Define the operator Λ by

$$\begin{array}{rccc} \Lambda : & F_0 & \rightarrow & F_0 \\ & f & \rightarrow & HH^* f. \end{array}$$

It follows from remark 1.4 that Λ is well defined and bounded. If F is the completion space of F_0 respectively to the norm $||.||_F$, then operator Λ has a unique extension, denoted also Λ, which is an isomorfism from the space F to its dual $F^{'}$.

Let's define the operator G by

$$\begin{array}{rccc} G : & L^2(0,T;X) & \rightarrow & L^2(-T,0;X) \\ & y & \rightarrow & y(T+.) \end{array}$$

Proposition 1.1 *Let* $x_0 \in X$ *and* $y_d \in L^2(0,T;X)$ *a given desired trajectory. If* $y_d \in G^{-1}(p_2(U(T)z_0)+F^{'})$, *where* $z_0 = (x_0, Fx_0)^T$, *then there exists a unique control* $u^* \in L^2(0,T;U)$ *such that*

$$\left\{ \begin{array}{lcl} x(., x_0, u^*) & = & y_d(.) \ on \ L^2(0,T;X) \\ ||u^*|| & = & min\{||v|| \ : \ x(., x_0, v) = y_d\}. \end{array} \right.$$

The input u^* *is given by*

$$u^* = H^* f, \tag{7}$$

where f *is the unique solution of the equation*

$$\Lambda f = G y_d - p_2(U(T)z_0). \tag{8}$$

Proof. Since $y_d \in G^{-1}(p_2(U(T)z_0) + F^{'}$, then

$$y_d(T+.) \in p_2(U(T)z_0) + F^{'}$$

but Λ is an isomorfism, hence there exists a unique $f \in F$ such that

$$y_d(T+.) = p_2(U(T)z_0) + \Lambda f.$$

Consider the control $u^* = H^* f$, we can write $y_d(T+.)$ as follows

$$y_d(T+.) = p_2(U(T)z_0) + Hu^*$$

which implies that $y_d(T+.) = p_2(z(T, x_0, u^*))$, and from remark 1.2 we deduce that $y_d(T+.) = y(T, x_0, u^*)$. Finally, from remark 1.1 we obtain

$$y_d = x(., x_0, u^*).$$

On the other hand, to proof the optimality of u^*, we consider the set

$$C = \{v \in L^2(0, T; U) : x(., x_0, v) = y_d\}.$$

For every $v \in C$, we have $x(., x_0, v) = x(., x_0, u^*)$. Hence $Hu^* = Hv$, consequently
$<< H(v - u^*), f >>= 0$, i.e., $< v - u^*, u^* >= 0$, thus $||u^*|| \leq ||v||$. ■

To complete the precedent result, we give a caracterization of the reachable trajectory.

Proposition 1.2 *Let* $W = \{x(., x_0, u) : u \in L^2(0, T; U)\}$ *be the set of all reachable trajectories, from an initial state* x_0*, on* $[0, T]$*, then*

$$W = G^{-1}(p_2(U(T)z_0) + F').$$

Proof. It follows from proposition 1.1 that

$$W \supset G^{-1}(p_2(U(T)z_0) + F').$$

Inversely, given $x(., x_0, u) \in W$, we consider the linear form

$$\begin{array}{rccl} \Psi: & F_0 & \rightarrow & I\!R \\ & f & \rightarrow & < Gx(., x_0, u) - p_2(U(T)z_0), f > . \end{array}$$

From the density of F_0 on F, we deduce that

$$|\Psi(f)| \leq ||u|| \, ||f|| \, , \; \forall f \in F.$$

Finally, it follows from the Riesz theorem that $Gx(., x_0, u) - p_2(U(T)z_0) \in F'$.
■

Remark 1.5 *Similarly to what was done by Emirsajlow in [5] and [6], the approach developped in this section can be used to resolve the following control problem*

$$min\{\int_0^T < u(t), Ru(t) > dt \; : \; u \in V_\alpha\},$$

where

$$V_\alpha = \{u \in L^2(0, T; U) \; : \; ||x(., x_0, u) - y_d||_{L^2(0,T;X)} \leq \alpha\} \; , \; (\alpha > 0)$$

and R *a self-adjoint definite positif operator.*

1.3 A finite dimensional case

Consider the system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t)\ ,\ t \in [0,T] \\ x(0) &= x_0, \end{cases} \tag{9}$$

where $A \in \mathcal{L}(I\!R^n), B \in \mathcal{L}(I\!R^n)$.

Proposition 1.3 *If there exists a matrice $K_2 \in \mathcal{L}(I\!R^n)$ such that the matrice BK_2 is invertible, then the space F_0 defined by (6) is given by*

$$F_0 = L^2(-T, 0; I\!R^n).$$

Proof. We take $y_d \in L^2(-T, 0; I\!R^n)$, $\epsilon > 0$ and seek a control u such that

$$||Hu - y_d|| \leq \epsilon.$$

Let K_1 be an arbitrary matrix in $\mathcal{L}(I\!R^n)$ and $h \in]0, T[$. We easily deduce from [7] that, under the invertibility hypothesis of the matrix BK_2, the system

$$\begin{cases} \dot{q}(t) &= Aq(t) + BK_1 q(t-h) + BK_2 v(t)\ ,\ t \in [0,T] \\ q(0) &= q_0 \\ q(r) &= \Phi_1(r)\ ,\ \forall r \in [-T, 0] \end{cases} \tag{10}$$

is approximately controllable on the space $M^2 = I\!R^n \times L^2(-T, 0; I\!R^n)$. In other words, for every $x_d = (a_d, b_d) \in M^2$, there exists a control v in $L^2(0, T; I\!R^n)$ such that

$$||(q(T, \Phi, v), q_T(., \Phi, v)) - x_d||_{M^2} \leq \epsilon, \tag{11}$$

where $\Phi = (q_0, \Phi_1) \in M^2$ is the initial state, $q(., \Phi, v)$ is the state variable corresponding to Φ and the input v, $q_T(., \Phi, v) \in L^2(-T, 0; I\!R^n)$ is defined by

$$[q_T(., \Phi, v)](r) = q(T + r, \Phi, v)\ ,\ r \in [-T, 0].$$

It follows from equation (11) that given an initial state $\Phi = (x_0, \Phi_1)$, where x_0 is the initial state of system 9 and $\Phi_1 \in L^2(-T, 0; I\!R^n)$, given a desired state $c_d = (x_d, y_d + p_2(U(T)z_0))$, where x_d is an arbitrary element of $I\!R^n$ and $z_0 = (x_0, Fx_0)$, there exists a control $v^* \in L^2(0, T; I\!R^n)$ such that

$$||(q(T, \Phi, v^*), q_T(., \Phi, v^*)) - c_d||_{M^2} \leq \epsilon,$$

hence

$$||q_T(., \Phi, v^*) - y_d - p_2(U(T)z_0)||_{L^2(-T,0;I\!R)} \leq \epsilon.$$

Then define the control variable

$$u_{v^*}(t) = K_1 q(t-h, \Phi, v^*) + K_2 v^*(t).$$

Since $q(., \Phi, v^*)$ is the solution of equation

$$\dot{q}(t) = Aq(t) + BK_1 q(t-h) + BK_2 v^*(t) \ , \ t \in [0,T],$$

we deduce that

$$\begin{cases} \dot{q}(t) & = & Aq(t,\Phi,v^*) + Bu_{v^*}(t) \ , \ t \in [0,T] \\ q(0,\Phi,v^*) & = & x_0, \end{cases} \tag{12}$$

thus

$$q(t,\Phi,v^*) = x(t,x_0,u_{v^*}) \ , \ r \in [0,T]$$

consequently,

$$q(T+r,\Phi,v^*) = x(T+r,x_0,u_{v^*}) \ , \ t \in [-T,0].$$

From remark 1.1, we have $q_T(.,\Phi,v^*) = y(T,x_0,u_{v^*})$, hence

$$||y(T,x_0,u_{v^*}) - y_d - p_2(U(T)z_0)||_{L^2(-T,0;I\!R)} \leq \epsilon.$$

Finally, from remark 1.2 we obtain that
$y(T,x_0,u_{v^*}) = p_2(z(T,x_0,u_{v^*}))$, and then $||Hu_{v^*} - y_d|| \leq \epsilon$. ■

2 A regional control trajectory problem

In this section we study the regional aspect of the control trajectory problem, i.e., we suppose that the state space is $X = L^2(\Omega)$, where Ω is an open bounded subset of $I\!R^n$, and we consider a region $\omega \subset \Omega$ and a desired trajectory $y_d \in L^2(0,T;L^2(\omega))$, then we investigate the control u solution of the problem

$$\begin{cases} [x(.,x_0,u)] & = & y_d \\ ||u|| & = & min\{||v|| : [x(.,x_0,v)]/\omega = y_d\}. \end{cases}$$

2.1 Definition of the problem-caracterization

Given a region ω of Ω, we consider the bounded operators M_ω, M and H_ω defined by

$$\begin{array}{rccl} M_\omega : & L^2(-T,0;X) & \rightarrow & L^2(-T,0;L^2(\omega)) \\ & f & \rightarrow & f/\omega \end{array}$$

$$\begin{array}{rccl} M : & Z = X \times Y & \rightarrow & L^2(-T,0;L^2(\omega)) \\ & (f,g) & \rightarrow & M_\omega(g) \end{array} \tag{13}$$

$$\begin{array}{rccl} H_\omega : & L^2(0,T;U) & \rightarrow & L^2(-T,0;L^2(\omega)) \\ & u & \rightarrow & M(\int_0^T U(T-r)Lu(r)dr). \end{array}$$

and the Hilbert space $E_0 = \overline{Im\, H_\omega} = (Ker\, H_\omega^*)^\perp$.

We define on $L^2(-T, 0; L^2(\omega))$, the semi norm

$$||f||_E = ||H_\omega^* f||_{L^2(0,T;U)}$$

and the corresponding inner product by

$$<< f, g >>_E = < H_\omega^* f, H_\omega^* g > \, , \; \forall f, g \in L^2(-T, 0; L^2(\omega)).$$

Remark 2.1 *We have*

(i) $||.||_E$ *is a norm on the space* E_0*.*

(ii) $(H_\omega H_\omega^*)(L^2(-T, 0; L^2(\omega))) \subset E_0$*.*

Define the operator Λ_ω by

$$\begin{array}{rccl} \Lambda_\omega : & E_0 & \rightarrow & E_0 \\ & f & \rightarrow & H_\omega H_\omega^* f. \end{array}$$

It follows from the precedent remark that Λ_ω is well defined, we also verify easily that it is bounded.

Let E be the completion space of E_0 relatively to the norm $||.||_E$. The operator Λ_ω can be extended continuously, and uniquely, to an isomorfirm defined from E to its dual E'. This extension is also denoted Λ_ω.

To establish the fundamental result of this section we introduce the operator G_ω defined by

$$\begin{array}{rccl} G_\omega : & L^2(0, T; L^2(\omega)) & \rightarrow & L^2(-T, 0; L^2(\omega)) \\ & y & \rightarrow & y(T + .) \end{array}$$

G_ω is bijectif and has an inverse operator described by

$$\begin{array}{rccl} G_\omega^{-1} : & L^2(-T, 0; L^2(\omega)) & \rightarrow & L^2(0, T; L^2(\omega)) \\ & y & \rightarrow & y(. - T) \end{array}$$

Proposition 2.1 *Let* $x_0 \in X$ *and* $y_d \in L^2(0, T; L^2(\omega))$ *a desired given trajectory,*
if $y_d \in G_\omega^{-1}(M(U(T)z_0) + E')$*, then there exists a unique control* $u^* \in L^2(0, T; U)$ *such that*

$$\left\{ \begin{array}{lcl} [x(., x_0, u^*)]/\omega & = & y_d(.) \text{ in } L^2(0, T; L^2(\omega)) \\ ||u^*|| & = & min\{||v|| : [x(., x_0, v)]/\omega = y_d\}. \end{array} \right.$$

u^* *is given by*

$$u^* = H_\omega^* f \tag{14}$$

where f *is the unique solution of equation*

$$\Lambda_\omega f = G_\omega y_d - M(U(T)z_0). \tag{15}$$

Moreover, the set $W_\omega = \{[x(., x_0, u)]/\omega : u \in L^2(0, T; U)\}$ *of all trajectories* $\omega - reachable$ *on* $[0, T]$ *is given by*

$$W_\omega = G_\omega^{-1}(M(U(T)z_0) + E').$$

Proof. The proof is similar to the ones of proposition 1.1 and 1.2. ∎

2.2 Application

Let ω and $\bar{\omega}$ be a given regions of Ω, $y_d \in L^2(\bar{\omega})$ a desired state and $z_d(.) \in L^2(0,T;L^2(\omega))$ a desired trajectory. The regional control trajectory problem consists of determining, under some hypothesis, the control u^* solution of following problem

$$||u^*|| \ = \ min\,||v||$$

where v verify

$$\begin{cases} x(T,x_0,v)/\bar{\omega} & = & y_d \\ x(.,x_0,v)/\omega & = & z_d(.) \text{ dans } L^2(0,T;X) \end{cases}$$

To resolve this problem, we define the following operators

$$\begin{array}{rcll} P: & X\times Y & \rightarrow & L^2(\bar{\omega}) \\ & (f,g) & \rightarrow & f/\bar{\omega}, \end{array}$$

and $H_{\bar{\omega},\omega} : L^2(0,T;U) \rightarrow L^2(\bar{\omega}) \times L^2(-T,0;L^2(\omega))$, such that

$$H_{\bar{\omega},\omega}(u) = (P(\int_0^T U(T-r)Lu(r)dr), M(\int_0^T U(T-r)Lu(r)dr))$$

where the operator M is defined by equation (13).

Consider the Hilbert space

$$N_0 = \overline{Im\, H_{\bar{\omega},\omega}} = (Ker\, H^*_{\bar{\omega},\omega})^\perp$$

and define on the space $L^2(\bar{\omega}) \times L^2(-T,0;L^2(\omega))$ the semi norm

$$||f||_N = ||H^*_{\bar{\omega},\omega} f||_{L^2(0,T;U)}.$$

Remark 2.2

i) $||.||_N$ *is a norm on the space* N_0.

ii) $(H_{\bar{\omega},\omega}H^*_{\bar{\omega},\omega})(L^2(\bar{\omega}) \times L^2(-T,0;L^2(\omega))) \subset N_0$.

We deduce from the above that the operator $\Lambda_{\bar{\omega},\omega}$ defined by

$$\begin{array}{rcll} \Lambda_{\bar{\omega},\omega}: & N_0 & \rightarrow & N_0 \\ & f & \rightarrow & (H_{\bar{\omega},\omega}H^*_{\bar{\omega},\omega})(f) \end{array}$$

is bounded and well defined.

Let N be the completion space of N_0 respectively to the norm $||.||_N$, The operator $\Lambda_{\bar{\omega},\omega}$ can be extended continuously and uniquely to an isomorfism defined from N to its dual space N'. This extension is also denoted by $\Lambda_{\bar{\omega},\omega}$.

Define the operator $K_{\bar{\omega},\omega}$ by

$$\begin{array}{rcll} K_{\bar{\omega},\omega}: & L^2(\bar{\omega}) \times L^2(-T,0;L^2(\omega)) & \rightarrow & L^2(\bar{\omega}) \times L^2(-T,0;L^2(\omega)) \\ & (y,z(.)) & \rightarrow & (y,z(.-T)). \end{array}$$

$K_{\bar{\omega},\omega}$ is bijectif and its inverse operator is given by

$$K^{-1}_{\bar{\omega},\omega}(y,z(.)) = (y,z(.-T)).$$

Proposition 2.2 *1) Let $x_0 \in X, y_d \in L^2(\bar{\omega})$ and $z_d \in L^2(0,T;L^2(\omega))$. If $(y_d, z_d) \in K^{-1}_{\bar{\omega},\omega}((P(U(T)z_0), M(U(T)z_0))+N')$, then there exists a unique control $u^* \in L^2(0,T;U)$ solution of the problem, and u^* is given by*

$$u^* = H^*_{\bar{\omega},\omega} f, \tag{16}$$

where f is the unique solution of the equation

$$\Lambda_{\bar{\omega},\omega} f = K_{\bar{\omega},\omega}(y_d, z_d) - (P(U(T)z_0), M(U(T)z_0). \tag{17}$$

2) The set

$$Q = \{([x(.,x_0,u)]/\bar{\omega}, [x(.,x_0,u)]/\omega) : u \in L^2(0,T;U)\}$$

is equal to the set

$$K^{-1}_{\bar{\omega},\omega}((P(U(T)z_0), M(U(T)z_0)) + N').$$

Proof. The proof is similar to the ones of propositions 1.1, 1.2. ■

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